# The Temperley-Lieb algebra at a root of unity 

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## Introduction

Section 1.1
Motivation
Let $\mathfrak{g l}_{r}$ be the matrix algebra over the complex numbers. The algebra $\mathfrak{g l}_{r}$ acts naturally from the left as a complex Lie algebra on a $r$-dimensional complex vector space $V$ with a chosen basis by left matrix multiplication. Moreover, this action induces a left action of $\mathrm{gl}_{r}$ on $V^{\otimes n}$, the $n$-fold tensor product of $V$. However, $V^{\otimes n}$ also admits a right $S_{n}$-module structure, where $S_{n}$ denotes the symmetric group in $n$ symbols, and secondly this right action commutes with the left action of $\mathfrak{g l}_{r}$. If $U\left(\mathfrak{g l}_{r}\right)$ denotes the universal enveloping algebra of $\mathfrak{g l _ { r }}$ and $\mathbb{C} S_{n}$ the group algebra of $S_{n}$, then this means that $V^{\otimes n}$ is a $U\left(\mathfrak{g l}_{r}\right)$ - $\mathbb{C} S_{n}$-bimodule. In fact, the $U\left(\mathfrak{g l}_{r}\right)$-left action and the $\mathbb{C} S_{n}$-right action do not only commute, the images of their actions in the endomorphism ring of $V^{\otimes n}$ are actually the centralizers of each other.

Now since $\mathbb{C} S_{n}$ is semisimple, the double centralizer theorem (cf. (Kna07, Theorem 2.43)) can be applied to obtain a decomposition of $V^{\otimes n}$ into summands of the form $V_{\lambda} \otimes S_{\lambda}$, where $V_{\lambda}$ is a simple $U\left(\mathrm{gl}_{r}\right)$-module and $S_{\lambda}$ is a simple $\mathbb{C} S_{n^{-}}$ module. This statement is known as the classical Schur-Weyl duality (cf. (Wey39)) and was first proven without using the double centralizer theorem.

However, there exist quantized versions of the algebras $U\left(\mathfrak{g l}_{r}\right)$ and $\mathbb{C} S_{n}$ : The complex quantum group $U_{q}\left(\mathrm{gl}_{r}\right)$ (see (HPK91, Section 0)) and the complex Hecke algebra $H_{n}(q)$ associated to the symmetric group $S_{n}$ (cf. (Hum90, Section 7.1)), where $q$ is a non-zero complex parameter. Moreover, if $H_{n}\left(q^{2}\right)$ is semisimple, i.e. if $q^{2}$ is not a $k$ th root of unity for $k=2, \ldots, n$ (see (HST15b, Theorem 5.1)), then one can quantize this Schur-Weyl story: The vector space $V^{\otimes n}$ admits a $U_{q}\left(\mathfrak{s l r}_{r}\right)-H_{n}\left(q^{2}\right)$ bimodule structure, such that the images of the actions of $U_{q}\left(\mathfrak{s l}_{r}\right)$ and $H_{n}\left(q^{2}\right)$ are the centralizers of each other (cf. (Jim86, Proposition 3)) and $V^{\otimes n}$ decomposes similar as in the classical case, now by using the double centralizer theorem, into summands $V_{\lambda} \otimes S_{\lambda}$, where $V_{\lambda}$ is a simple $U_{q}\left(\mathrm{gl}_{r}\right)$-module and $S_{\lambda}$ is a simple $H_{n}\left(q^{2}\right)$ module. This statement is the quantized version of Schur-Weyl duality for $U_{q}\left(\mathfrak{s l}_{r}\right)$ and $H_{n}\left(q^{2}\right)$.

Identifying $V^{8 n}$ as $q$-tensor space and using (Mur95, Theorem 6.3) and (Mur95, Theorem 7.2) or using (DJ86, Corollary 4.12) actually implies for $n \geq r$, that the Hecke algebra $H_{n}\left(q^{2}\right)$ does not act faithfully on $V^{\otimes n}$; in particular, the image of its action beeing the centralizer of the quantum group $U_{q}\left(\mathrm{gII}_{r}\right)$ is a proper quotient of $H_{n}\left(q^{2}\right)$. In the case where $r=2$, this quotient turns out to be another known object, namely the Temperley-Lieb algebra $T L_{n}(q)$, which is a diagram algebra of so-called "planar Brauer diagrams", see also (Jon85). We are primarly interested in the study of this algebra, the Temperley-Lieb algebra $T L_{n}(q)$. Therefore, by examining the kernel of the $H_{n}\left(q^{2}\right)$-action, one can describe the representation theory of the Temperley-Lieb algebra via the Hecke algebra, supposed that $H_{n}\left(q^{2}\right)$ is semisimple, where $q$-Schur-Weyl duality holds.

However the Temperley-Lieb algebra $T L_{n}(q)$ can be studied intrinsically with-
out using $q$-Schur-Weyl duality. If $q^{2}$ is a $k$ th root of unity for a $k \leq n$, then $T L_{n}(q)$ is not semisimple anymore (cf. (HST15a, Proposition 5.1)), neither is $H_{n}\left(q^{2}\right)$ (see (HST15b, Theorem 5.1)) and nor can $q$-Schur-Weyl duality be applied. Nevertheless, we have some tools at hand to examine the non-semisimple Temperley-Lieb algebra. In the semisimple case, decomposing $T L_{n}(q)$ into simple $T L_{n}$-modules is equivalent to describe all minimal central idempotents $T L_{n}(q)$. Therefore, a good approach would be to describe the minimal central idempotents, also called (higher order) Jones-Wenzl projectors (cf. (Wen87),(GW93) and (CH15)), by explicit formulas and try to move as many of these formulas as possible to the non-semisimple world. In (CH15) these Jones-Wenzl projectors are described with aid of a complete set of pairwise orthogonal minimal idempotents $p_{t}$ in $T L_{n}(q)$, which can be indexed in a pretty way by paths $t$ in the branching graph of $T L_{n}(q)$, supposed that $T L_{n}(q)$ is semisimple. However, if $T L_{n}(q)$ is not semisimple, then these minimal idempotents are not all well-defined and hence another complete set of idempotents is needed to describe minimal central idempotents. In (GW93) a sufficient number of well-defined idempotents in the non-semisimple $T L_{n}(q)$ is found along with a description of the minimal central idempotents modulo the radical in $T L_{n}(q)$.

Since the Temperley-Lieb algebra is an example of a diagram algebra, the Jones-Wenzl projectors and also the idempotents presented in (GW93) can be expressed by diagramatic language. However, the arguments presented in (GW93) are of pure algebraic nature, so our main interest is to translate all the proofs of (GW93) into diagramatic language. When dealing with the diagramatic presentation of the Temperley-Lieb algbera, one soon encounters a certain basis consisting of arc diagrams, which is an example of a cellular basis (cf. (GL96)) equipping the algebra $T L_{n}(q)$ with a cellular structure. If $T L_{n}(q)$ is semisimple, then the complete set of pairwise orthogonal minimal idempotents $p_{t}$ gives rise to a basis $p_{t, s}$ of elements in non-trivial subspaces $p_{t} T L_{n}(q) p_{s}$ and a natural question would be to relate these elements $p_{t, s}$ to the cellular basis consisting of arc diagrams. This relation turns out to be an upper triangular base change, which we believe was not known yet. As a side result, a partial coefficient formula for the minimal idempotents $p_{t}$ expressed in the cellular basis consisting of arc diagrams is obtained.

## Section 1.2

## Overview

This master thesis is seperated into three sections.
Section 2 is about recalling the definition and commonly known facts of the complex Temperley-Lieb algebra $T L_{n}(q)$ and the complex Hecke algebra $H_{n}(q)$, both depending on a natural number $n$ and a complex non-zero parameter $q$. In Section 2.1 these definitions are stated and moreover the algebra $T L_{n}(q)$ is identified as a quotient of $H_{n}\left(q^{2}\right)$.

Two important notions involved in describing the algebras $T L_{n}(q)$ and $H_{n}(q)$ are those of partitions and tableaux. Section 2.2 starts by describing a cellular basis of $T L_{n}(q)$ consisting of arc diagrams $\beta_{t, s}$ indexed by pairs of tableaux $t$ and $s$ of same shape $\lambda$, where $\lambda$ ranges over $\operatorname{Par}_{2}(n)$, the set of partitions of $n$ with at
most 2 rows. Partitions and tableaux are also used to describe the simple modules of the algebras $T L_{n}(q)$ and $H_{n}(q)$, in case they are semisimple: The set of simple modules $S_{\lambda}$ of $H_{n}(q)$ (resp. $\left.T L_{n}(q)\right)$ is parametrized by $\operatorname{Par}(n)$, the set of partitions of $n$ (resp. $\left.\operatorname{Par}_{2}(n)\right) . S_{\lambda}$ has a basis consisting of tableaux $t$ of shape $\lambda$, such that the algebra actions can be described by explicit formulas. Moreover, the module structures are compatible with the identification of $T L_{n}(q)$ as a quotient of $H_{n}\left(q^{2}\right)$.

Section 2 ends with Section 2.3, which is devoted to sketch a proof of a quantized version of Schur-Weyl duality, which in turn yields another description of the Temperley-Lieb algbera. If $V$ is a complex $r$-dimensional vector space with a chosen basis, then it can be equipped with an $U_{q}\left(\mathfrak{g l}_{r}\right)$-module structure. It actually suffices to consider the quantum-group $U_{q}\left(\mathfrak{s l}_{r}\right)$, since the image of its action coincides with that of $U_{q}\left(\mathfrak{g I}_{r}\right)$. Using the comultiplication of $U_{q}\left(\mathfrak{s l}_{r}\right)$, the $n$-fold tensor product $V^{\otimes n}$, becomes a $U_{q}\left(\mathfrak{s l}_{r}\right)$-module. Moreover, $V^{\otimes n}$ also admits a right $H_{n}\left(q^{2}\right)$-module structure commuting with the left $U_{q}\left(\mathfrak{s l}_{r}\right)$-action. Now if $H_{n}\left(q^{2}\right)$ is semisimple, $q$-Schur-Weyl duality decomposes $V^{\otimes n}$ into summands of the form $V_{\lambda} \otimes S_{\lambda}$, where $V_{\lambda}$ is a simple $U_{q}\left(\mathfrak{s l}_{r}\right)$-module, $S_{\lambda}$ a simple $H_{n}\left(q^{2}\right)$-module and $\lambda$ ranges over $\operatorname{Par}_{r}(n)$, the set of partitions of $n$ with at most $r$ rows. Analyzing the kernel of the $H_{n}\left(q^{2}\right)$-action on $V^{\otimes n}$ shows for $r=2$, that the Temperley-Lieb algebra $T L_{n}(q)$ is isomorphic to the centralizer $\operatorname{End}_{U_{q}\left(\mathfrak{s l}_{2}\right)}\left(V^{\otimes n}\right)$ of $U_{q}\left(\mathfrak{s l}_{2}\right)$ in the endomorphism ring of $V^{\otimes n}$.

Section 3 is the core of this master thesis, we mainly follow (GW93). Our presentation differs slightly, we tried to connect (GW93) with the more modern diagramatic presentations of the Temperley-Lieb algebra. To understand the nonsemisimple algebra $T L_{n}(q)$, we follow the "evaluation principle", meaning that we show structure results for semisimple generic algebra $T L_{n}(v)$ and deduce by "evaluating" at $v=q$ corresponding results for the non-semisimple version. Section 3 is therefore devided into two parts, the first is devoted to the generic version and the second to the non-semisimple one. However, we stress that all statements in Section 3 only use diagramatic arguments.

Section 3.1 starts in Section 3.1.1 by decomposing $T L_{n}(v)$ with aid of $q$-SchurWeyl duality into summands of the form $\operatorname{End}_{U_{v}\left(\mathrm{~s}_{2}\right)}\left(S_{\lambda} \otimes V_{\lambda}\right) \cong \operatorname{End}_{\mathbb{C}}\left(S_{\lambda}\right)$. This leads to a shift of perspective, instead of studying simple $T L_{n}(v)$-modules, the focus is turned onto the minimal central idemptents $z_{\lambda} \in T L_{n}(v)$ corresponding to the identity in $\operatorname{End}\left(S_{\lambda}\right)$. The special case $\lambda=(n)$ is the first case to treat: $z_{(n)}$ turns out to be the $n$th Jones-Wenzl projector $p_{n} \in T L_{n}(v)$, which can be defined by a diagramatic recursive rule. Following (CH15), we continue in Section 3.1.2 by introducing path idempotents $p_{t}$ for $t$ a standard tableau of shape $\lambda$ in $\operatorname{Par}_{2}(n)$. The path idempotents are also defined by a recursive diagramatic rule and form a complete set of pairwise orthogonal minimal idempotents in $T L_{n}(v)$. Moreover their action on $S_{\lambda}$ is particularly simple: $p_{t}$ fixes the basis element $t \in S_{\lambda}$ and sends every other basis element $s \in S_{\mu}$ for arbitrary $\mu$ to zero.

Motivated by the action of the path idempotents on simple modules, the higher order Jones-Wenzl projectors $p_{n, k}$ are defined in Section 3.1.3 by diagramatic language, such that they correspond to the minimal central idempotents $z_{\lambda}$ : For $\lambda$ in
$\operatorname{Par}_{2}(n)$ with $k=\lambda_{1}-\lambda_{2}$, the $k$ th higher order Jones-Wenzl projector $p_{n, k}$ is defined to be the sum of all path idempotents $p_{t}$ such that $t$ is of shape $\lambda$.

To push formulas concerning elements in the generic algebra $T L_{n}(v)$ to the non-semisimple complex version $T L_{n}(q)$, where $q$ is a $2 l$ th root of unity, we start Section 3.2 by clarifying, which elements of the generic Temperley-Lieb algebra actually give rise to elements in the complex non-semisimple version. These elements are called evaluable and are roughly speaking linear combinations of arc diagrams, where all coefficients are elements in $\mathbb{C}(v)$ without a pole in $q$. Not all path idempotens $p_{t}$ turn out to be evaluable, but certain sums of those are. The involved notions are:

- The $m$ th critical line consist of all partitions $v$, such that $v_{1}-v_{2}+1=m l$. Such a partition is called critical and so are its tableaux.
- If $r$ is the maximal critical subtableau of $t$ ending on the $m$ th critical line, then $\bar{t}$ is defined to be the path obtained from $t$ by reflecting $t \backslash r$ in the branching graph about that $m$ th critical line.

In Section 3.2.1 we try to determine which $p_{[t]}:=p_{t}+p_{\bar{t}}$ are evaluable and moreover we construct new evaluable idempotents (not necessarily path idempotents) out of old ones. This contains a couple of rather calculation heavy arguments, which are needed for Section 3.2.2. The former now contains the main results of this master thesis, including a description of the maximal semisimple quotient and the radical of $T L_{n}(q)$. To understand the main statement, the following terminology is needed.

- For $\lambda$ in $\operatorname{Par}_{2}(n)$, $[\lambda]$ denotes the orbit of $\lambda$ under the action of the reflection group of $\mathbb{Z}$ acting on $\operatorname{Par}_{2}(n)$ by reflecting about critical lines in the branching graph. Two partitions $\mu$ and $v$ in $[\lambda]$ are called adjacent, if there is exactly one critical line between $\mu$ and $\nu$.
- For a non-critical partition $\lambda \in \operatorname{Par}_{2}(n)$ between the $m$ th and the $m+1$ th critical lines, $L(\lambda)$ is defined to be the set of tableaux of shape $\lambda$ with proper maximal critical subtableau on the $m$ th critical line.
- If $\lambda$ is critical, then $f_{\lambda}$ is the number of tableaux of shape $\lambda$. On the other hand, for $\lambda$ non-critical, $z_{\lambda}^{L}$ is the sum over all $p_{[t]}$, where $t$ ranges over $L(\lambda)$, and $f_{\lambda}^{L}$ denotes the cardinality of $L(\lambda)$. Furthermore $z_{[\lambda]}$ is defined to be the sum of all $z_{v}^{L}$, where $v$ is ranging over [ $\left.\lambda\right]$.

The results of Section 3.2.2 are summarized be the following theorem (see Theorem 3.2.28 and Theorem 3.2.30):

Theorem ((GW93)).

1. If $\lambda$ is critical or to the left of the first critical line, then $z_{\lambda}(q)=z_{\lambda}^{L}(q)$ is a minimal central idempotent in $T L_{n}(q)$. Furthermore $z_{\lambda} T L_{n}(q) \cong M_{f_{\lambda}} \mathbb{C}$.
2. If $\lambda$ is non-critical and to the right of the first critical line, then $z_{\lambda}^{L}$ is evaluable and $z_{\lambda}^{L} T L_{n} z_{\lambda}^{L}(q) \cong\left\{\left(\begin{array}{cc}A & B \\ 0 & A\end{array}\right), A, B \in M_{f_{\lambda}^{L}} \mathbb{C}\right\}$.
Moreover, $z_{[\lambda]}(q)$ is a minimal central idempotent. The radical of $z_{[\lambda]} T L_{n}(q)$ is nilpotent of order 3 and spanned by the spaces $z_{\mu}^{L} T L_{n} z_{v}^{L}(q)$ for adjacent diagrams $\mu, v$ in $[\lambda]$ and by the algebras $\operatorname{rad}\left(z_{\mu}^{L} T L_{n} z_{\mu}^{L}(q)\right)$ for $\mu \in[\lambda]$. The maximal semisimple quotient of $z_{[\lambda]} T L_{n}(q)$ is isomorphic to $\bigoplus_{\mu \in[\lambda]} M_{f_{\mu}^{L}} \mathbb{C}$.

Section 4 is about consequences of Section 3.1. Since the $p_{t}$ form a complete set of pairwise orthogonal idempotents, $T L_{n}(v)$ decomposes into subspaces of the form $p_{t} T L_{n}(v) p_{s}$, which are all at most one-dimensional. In particular, there exists a basis indexed by pairs of tableaux $s$ and $t$ of elements $p_{t, s}$ in the non-zero subspaces $p_{t} T L_{n}(v) p_{s}$. In fact, $p_{t} T L_{n}(v) p_{s}$ is non-zero if and only if $s$ and $t$ are of same shape, thus the set of elements $p_{t, s}$ indexed by pairs of tableaux of same shape is a basis of $T L_{n}(v)$. In Section 3.2.1 this basis was already implicitly used, however it was not explicitly defined. The aim of Section 4.1 is to define this basis in a consistent way, such that it satisfies the following properties:

- If $t=s$, then $p_{t, t}$ coincides with the path idempotent $p_{t}$ (see Definition 4.1.8).
- If $t, s$ and $r$ are of same shape, then $p_{t, s} p_{s, r}=p_{t, r}$ (see Proposition 4.1.10).

In Section 4.2, the basis for $T L_{4}(v)$ is expressed in the basis $\beta_{t, s}$ by explicit calculation. If one orders both bases by a well-known partial order, the dominance order, the matrix expressing the base change turns out to be upper triangular. We show in Section 4.3, that this is no coincidence, but that in general the basis of elements $p_{t, s}$ and that of elements $\beta_{t, s}$ are in an upper triangular relation. As a side result, a partial coefficient formula is obtained. The results of Section 4.3 can be summarized by the following theorem (see Theorem 4.3.18):

Theorem. The basis $p_{t, s}$ is related by an upper triangular relation with respect to the dominance order to the cellular basis $\beta_{t, s}$ consisting of arc diagrams. Moreover for $u, w, t$, s of same shape, the coefficent $c_{u, w}^{t, s}$ of $\beta_{u, w}$ in $p_{t, s}=\sum_{u \unlhd t, w \unlhd s} c_{u, w}^{t, s} \beta_{t, s}$ can be described by an inductive formula.

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## The Temperley-Lieb algebra

Let $q \in \mathbb{C}^{\times}$be a complex number. In Section 2.1 the definition of the main object of this thesis is recalled, the complex Temperley-Lieb algebra $T L_{n}(q)$ in $n$ strands, which is a unital associative $\mathbb{C}$-algebra depending on the parameter $q$. To understand its representation theory and to find out, when it is semisimple, it is convenient to consider another well-known algebra, the Hecke algebra $H_{n}(q)$ of type A. Although the reader might know these algebras, we summarize in Section 2.2 known facts about the representation theory of $T L_{n}(q)$ and of $H_{n}(q)$ and moreover, we get to know for which $q$ they are semisimple: $T L_{n}(q)$ is semisimple if $q^{2}$ is not a $k$ th root of unity for $k=2, \ldots, n$ and $H_{n}(q)$ is semisimple if $q$ is not a $k$ th root of unity for $k=2, \ldots, n$. Semisimplicity allows to apply another powerful tool, the double centralizer theorem, which results in a version of a Schur-Weyl duality in Section 2.3. Without semisimplicity a more detailed study is necessary, this aspect is treated in more detail in Section 3 and excluded in this section.

All the results in this section are already known, but we believe that it is useful to have them combined at hand, instead of citing them only.

## Section 2.1 The Hecke algebra and the Temperley-Lieb algebra

In this section $q$ will always denote an element in $\mathbb{C}^{\times}=\mathbb{C} \backslash\{0\} . v$ on the other hand will always denote a generic parameter.

Although one can define the Hecke algebra for any Coxeter system, see for example (Hum90, Section 7.1), we only state the definition of the Hecke algebra associated to the Coxeter system $S=\left\{s_{1}, \ldots, s_{n-1}\right\} \subset S_{n}$ of simple transpositions $s_{i}$ in the symmetric group $S_{n}$; this is the Hecke algebra of type $A_{n-1}$, which we will call in this thesis the Hecke algebra in $n$-strands or just the Hecke algebra.
Definition 2.1.1. 1. The generic Hecke algebra in n-strands over $\mathbb{Z}\left[v, v^{-1}\right]$ is defined to be the unital, associative $\mathbb{Z}\left[v, v^{-1}\right]$-algebra $H_{n}^{\mathbb{Z}}(v)$ generated by $T_{1}, \ldots, T_{n-1}$ subject to the following relations:

$$
\begin{align*}
T_{i}^{2} & =(v-1) T_{i}+v, & & \text { if } 1 \leq i \leq n-1,  \tag{2.1}\\
T_{i} T_{i+1} T_{i} & =T_{i+1} T_{i} T_{i+1}, & & \text { if } 1 \leq i \leq n-2,  \tag{2.2}\\
T_{i} T_{j} & =T_{j} T_{i}, & & \text { if }|i-j|>1 . \tag{2.3}
\end{align*}
$$

2. Moreover, the generic Hecke algebra in n-strands over $\mathbb{C}(v)$ is defined by setting $H_{n}(v):=H_{n}^{\mathbb{C}}(v):=H_{n}^{\mathbb{Z}}(v) \otimes_{\mathbb{Z}} \mathbb{C}(v)$.
3. Finally, for $q \in \mathbb{C}^{\times}$, the complex Hecke algebra in $n$-strands is defined to be $H_{n}(q):=H_{n}^{\mathbb{C}}(q):=H_{n}^{\mathbb{Z}}(v) \otimes_{\mathbb{Z}\left[v, v^{-1}\right]} \mathbb{C}$, where $v$ acts on $\mathbb{C}$ by substitution.
One sees easily, that the generators $T_{i}$ are invertible with inverse $T_{r}^{-1}=q^{-1}\left(T_{r}-\right.$ $q+1)$. Now, for $w \in S_{n}$, the element $T_{w}$ is defined by

$$
T_{w}:=T_{i_{1}} \ldots T_{i_{r}} \in H_{n}^{\mathbb{Z}}(q)
$$

where $w=s_{i_{1}} \ldots s_{i_{r}}$ is a reduced espression for $w$. By (2.2) and (2.3) this is independant of the choice of the chosen reduced expression and moreover, the following proposition, which can be found in (Hum90, Section 7.1), identifies the set of elements $T_{w}, w \in S_{n}$ as a basis of $H_{n}^{\mathbb{Z}}(q)$ :

Proposition 2.1.2. The set $\left\{T_{w}, w \in S_{n}\right\}$ is a basis of the Hecke algebra $H_{n}^{\mathbb{Z}}(q)$.
For $d \in \mathbb{Z}_{\geq 1}$ and $q \neq 1$, the notation of the usual quantum integers, defined by

$$
[d]=[d]_{q}=q^{d-1}+q^{d-3}+\cdots+q^{-d+1}=\frac{q^{d}-q^{-d}}{q-q^{-1}} \in \mathbb{Z}\left[q, q^{-1}\right]
$$

is used. The same goes for $v$. The main object of study of this thesis is introduced:

## Definition 2.1.3. 1 . The generic Temperley-Lieb algebra in n-strands over

 $\mathbb{Z}\left[v, v^{-1}\right]$ is the unital, associative $\mathbb{Z}\left[v, v^{-1}\right]$-algebra $T L_{n}^{\mathbb{Z}}(v)$ generated by $U_{1}, \ldots, U_{n-1}$ with relations$$
\begin{align*}
U_{i}^{2} & =[2]_{v} U_{i}, & & \text { if } 1 \leq i \leq n-1,  \tag{2.4}\\
U_{i} U_{j} U_{i} & =U_{i}, & & \text { if }|i-j|=1,  \tag{2.5}\\
U_{i} U_{j} & =U_{j} U_{i}, & & \text { if }|i-j|>1 . \tag{2.6}
\end{align*}
$$

2. Secondly, the generic Temperley-Lieb algebra in n-strands over $\mathbb{C}(v)$ is defined to be $T L_{n}(v):=T L_{n}^{\mathbb{C}}(v):=T L_{n}^{\mathbb{Z}}(v) \otimes_{\mathbb{Z}\left[v, v^{-1}\right]} \mathbb{C}(v)$.
3. Moreover, for $q \in \mathbb{C}^{\times}$the complex Temperley-Lieb algebra in $n$-strands is defined by $T L_{n}(q):=T L_{n}^{\mathbb{C}}(q):=T L_{n}^{\mathbb{Z}}(v) \otimes_{\mathbb{Z}\left[v, v^{-1}\right]} \mathbb{C}$, where $v \in \mathbb{Z}\left[v, v^{-1}\right]$ acts on $\mathbb{C}$ by substitution.

For the rest of this section, only the complex versions of the Hecke algebra and the Temperley-Lieb algebra are needed, the generic version will return Section 3. However, the theory for $T L_{n}(q)$, where $q$ is not a root of unity, is the same as for $T L_{n}(v)$. The following theorem establishs a connection between $T L_{n}(q)$ and $H_{n}\left(q^{2}\right)$ :

Theorem 2.1.4. The maps

$$
\begin{align*}
& \phi_{1}: H_{n}\left(q^{2}\right) \rightarrow T L_{n}(q), T_{i} \mapsto q U_{i}-1,  \tag{2.7}\\
& \phi_{2}: H_{n}\left(q^{2}\right) \rightarrow T L_{n}(q), T_{i} \mapsto-q U_{i}+q^{2} \tag{2.8}
\end{align*}
$$

are surjective morphisms of algebras. Moreover $\operatorname{ker} \phi_{1}$ is generated by

$$
T_{1} T_{2} T_{1}+T_{1} T_{2}+T_{2} T_{1}+T_{1}+T_{2}+1
$$

and $\operatorname{ker} \phi_{2}$ by $q^{-6} T_{1} T_{2} T_{1}-q^{-4} T_{1} T_{2}-q^{-4} T_{2} T_{1}+q^{-2} T_{1}+q^{-2} T_{2}-1$.
Proof. It is easy to check that $\phi_{i}$ respects the relations and defines a surjective morphism of unital algebras. That the element is in the kernel can be checked by direct computation. That it spans the kernel follows by analyzing the idempotents in the Hecke algebra in the kernel of the action induced by Schur-Weyl duality. However, we do not have a good reference and refer to (Wee12, Section 3.2).

In the following pages, we will use the presentation of $T L_{n}(q)$ as a quotient of $H_{n}\left(q^{2}\right)$, however we choose to work with $\phi_{1}$ instead of $\phi_{2}$, to stay compatible with the notions in (GW93), which is the main source of Section 3. But this choice does actually not amount to greater disadvantage:

Remark 2.1.5. $\phi_{1}$ and $\phi_{2}$ are related via $\phi_{1}=\alpha \circ \phi_{2}$ and $\phi_{2}=\phi_{2} \circ \beta$, where

$$
\begin{aligned}
\alpha: T L_{n}(q) & \rightarrow T L_{n}(q), U_{i} \mapsto-U_{i}+[2] \\
\beta: H_{n}(q) & \rightarrow H_{n}(q), T_{i} \mapsto-T_{i}+q^{2}-1
\end{aligned}
$$

It is commonly known that the Temperley-Lieb algebra $T L_{n}^{\mathbb{Z}}(q)$ has a $\mathbb{Z}\left[q, q^{-1}\right]$ basis consisting of "planar Brauer diagrams", here called arc diagrams, in $2 n$ points, where we follow (GL96, Example 1.4).

These consists of two edges, called top and bottom edge, each of them endowed with $n$ vertices, such that each vertex is joined to just one another vertex and none of the joins intersect, when drawn in the rectangle defined by the two edges. Multiplication of diagrams is given by vertical juxtaposition, removing interior circles and multiplying with the factor [2] for each removed interior circle. The generators $U_{i}$ and the unit element correspond to the arc diagrams in Figure 1. We also call a vertex a bottom vertex, if it is on the bottom edge, and a top vertex, if it is on the top edge. In the same spirit, $T L_{n, d}(q)$ is the space of arc diagrams


Figure 1: The generator $U_{i}$ and the identity.
with $n$ vertices on the top edge and $d$ vertices on the bottom edge, supposed that $d$ and $n$ are of same pairity. This is then a $T L_{n}(q)-T L_{d}(q)$-bimodule, where the actions are defined by concatenation of arc diagrams, removing interior circles and multiplying with [2] for each removed circle, similar as before. Moreover, stacking diagrams defines a $T L_{n}(q)-T L_{m}(q)$-bimodule morphism

$$
T L_{n, d}(q) \otimes T L_{d, m}(q) \rightarrow T L_{n, m}(q)
$$

The representation theory for $T L_{n}(q)$, if $q \in \mathbb{C}^{\times}$is not a root of unity, is well understood, as well as that for $H_{n}(q)$. The main definitions and facts concerning this are stated in the next section, namely Section 2.2.

## Section 2.2

Combinatorics and representations
To understand this section, the reader is assumed to be familiar with the notion of partitions and tableaux. The set of partitions of $n$ is denoted by $\operatorname{Par}(n)$ and $\operatorname{Par}_{r}(n)$
describes the subset of those with at most $r$ rows. For a partition $\lambda \in \operatorname{Par}(n)$,

$$
\langle\lambda\rangle=\left\{(i, j) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}, 1 \leq j \leq \lambda_{i}, 1 \leq i \leq h\right\}
$$

denotes the associated Young diagram. Its elements are called nodes. Moreover the graph $B$ is defined to be the graph with vertex set $\cup_{n} \operatorname{Par}(n)$ and oriented edges $\lambda \rightarrow \mu$, whenever $\langle\mu\rangle$ can be obtained from $\langle\lambda\rangle$ by adding a node. Consequently $B_{2}$ denotes the subgraph induced by $\bigcup_{n} \operatorname{Par}_{2}(n)$, see also Figure 2. The orientation in Figure 2 is implicitly set from top to bottom. It will turn out later in this section (see Corollary 2.2.12), that this graph is the branching graph for the TemperleyLieb algebras $T L_{n}(q), n \in \mathbb{N}$.


Figure 2: The branching graph $B_{2}$ of the algebras $T L_{n}, n \in \mathbb{N}$.
Following standard notation, $\operatorname{Tab}(\lambda)$ denotes the set of tableaux of shape $\lambda \in$ $\operatorname{Par}(n)$ and $\operatorname{Std}(\lambda)$ the subset of standard ones, where we mean by standard, strictly row-increasing and strictly column-increasing. Similarly, the set of all tableaux is denoted by $\operatorname{Tab}(n)=\bigcup_{\lambda \in \operatorname{Par}(n)} \operatorname{Tab}(\lambda)$ and that of all standard ones by $\operatorname{Std}(n)=$ $\bigcup_{\lambda \in \operatorname{Par}(n)} \operatorname{Std}(n)$. In the same spirit, we use the notation $\operatorname{Tab}_{r}(n)$ and $\operatorname{Std}_{r}(n)$.

Furthermore, we can identify a tableau $t \in \operatorname{Tab}(\lambda)$ with a bijection $\langle\lambda\rangle \rightarrow$ $\{1, \ldots, n\}$. We also write $\operatorname{Shape}(t)=\lambda$. Moreover, if $t$ is standard, then $t$ can also be identified with a path $\emptyset \rightarrow \lambda^{(1)} \rightarrow \cdots \rightarrow \lambda^{(n)}=\lambda$ in $B$. Respecting this, $t^{\prime}$ denotes the subpath obtained from $t$ by removing $\lambda^{(n)}$. If $\lambda \in \operatorname{Par}_{2}(n)$, then $t$ corresponds actually to a path in $B_{2}$, therefore, if this is the case, we denote by $t^{+}$
and $t^{-}$the two possible extensions of $t$ in $B_{2}$, where $t^{+}$is the extension obtained by adding $n+1$ to the first row of $t$ and $t^{-}$by adding $n+1$ to the second line $\left(t^{-}\right.$does not exist if $\lambda_{2}=\lambda_{1}$ ).

There is a partial order on $\operatorname{Par}(n)$, which extends to $\operatorname{Std}(n)$ and which is usually called the dominance order:

Definition 2.2.1. 1. For $\mu, \lambda \in \operatorname{Par}(n)$, $\lambda$ dominates $\mu$, written $\lambda \unrhd \mu$, if

$$
\lambda_{1}+\cdots+\lambda_{j} \geq \mu_{1}+\cdots+\mu_{j} \quad \text { for all } j \geq 1
$$

2. Moreover, for $t, s \in \operatorname{Std}(n)$, $t$ is said to dominate $s$, again written $t \unrhd s$, if

$$
\operatorname{Shape}\left(\left.t\right|_{k}\right) \unrhd \operatorname{Shape}\left(\left.s\right|_{k}\right) \quad \text { for all } k=1, \ldots, n \text {, }
$$

where $\left.t\right|_{k}$ is obtained from $t:\langle\lambda\rangle \rightarrow\{1, \ldots, n\}$ by restricting to $\{1, \ldots, k\}$.
Remark 2.2.2. 1. Let $t^{\lambda} \in \operatorname{Std}(\lambda)$ be a tableau, such that $t^{\lambda} \unrhd t$ for all $t \in$ $\operatorname{Std}(\lambda) \cdot t^{\lambda}$ is unique and $t^{\lambda}$ has its numbers $1, \ldots, n$ ordered from left to right, top to bottom, rows before columns.
2. When dealing with two-row partitions and the corresponding tableaux, one can easily visualize the dominance order in the branching graph: For two partitions $\lambda=\left(\lambda_{1}, \lambda_{2}\right), \mu=\left(\mu_{1}, \mu_{2}\right) \in \operatorname{Par}_{2}(n)$ it is clear that $\lambda \unrhd \mu$, if $\lambda_{1} \geq \mu_{1}$. Moreover two paths $t, s \in \operatorname{Std}(n)$ satisfy $t \unrhd s$ if and only if $t$ is weakly to the right of $s$. An example is given in Figure 3.

With the dominance order defined, it is now possible to determine the dimension of $H_{n}(q)$. To do so, we observe first that

$$
\begin{equation*}
H_{n}(q) \rightarrow H_{n}(q), T_{w} \mapsto T_{w}^{*}=T_{w^{-1}} \tag{2.9}
\end{equation*}
$$

defines an anti-automorphism of algebras. Moreover, the symmetric group $S_{n}$ acts on the set of tableaux $\operatorname{Tab}(\lambda)$ of shape $\lambda \in \operatorname{Par}(n)$ by permuting entries. Writing $S_{\lambda} \subset S_{n}$ for the row stabilizer of $t^{\lambda}$, we can define for $\lambda \in \operatorname{Par}(n)$ and $t, s \in \operatorname{Std}(\lambda)$,

$$
\begin{equation*}
x_{\lambda}:=\sum_{w \in S_{\lambda}} T_{w} \text { and } x_{t, s}^{\lambda}:=T_{d(s)}^{*} x_{\lambda} T_{d(t)}, \tag{2.10}
\end{equation*}
$$

where $d(t) \in S_{n}$ is a reduced element, such that $t=d(t)\left(t^{\lambda}\right)$. Then one can prove the following theorem, see (Mur95, Theorem 4.17).

Theorem 2.2.3. The set $\left\{x_{t, s}^{\lambda}, t, s \in \operatorname{Std}(\lambda), \lambda \in \operatorname{Par}(n)\right\}$ is a basis of $H_{n}(q)$. In particular $\operatorname{dim} H_{n}(q)=n!=\# S_{n}$.

This basis is also called Murphy's standard basis. Having obtained the dimension of the Hecke algebra, it would also be nice to do so for $T L_{n}(q)$. Let $x \mapsto \tilde{x}$ denote the map induced by flipping diagrams vertically in $T L_{n}(q)$.



Figure 3: Two paths $s \triangleleft t$ and two non-compatible paths $w$ and $u$.

Definition 2.2.4. For an arc diagram $\beta$, a line of $\beta$ is horizontal, if its endpoints are both top or bottom vertices, otherwise it is vertical. To a path tof shape $\lambda \in \operatorname{Par}_{2}(n)$ with $d=\lambda_{1}-\lambda_{2}$, we associate an arc diagram $\beta_{t, \text {, }} \in T L_{n, d}(q)$ by the condition:

- $k \in\{1, \ldots, n\}$ is in the second row of $t$ if and only if the $k$ th top vertex is the right endpoint of a horizontal line.

Moreover, for $\lambda \in \operatorname{Par}_{2}(n)$ and $s, t \in \operatorname{Std}(\lambda)$ we also write $\beta_{s, t}=\beta_{s, \cdot} \cdot \tilde{\beta}_{t, \cdot} \in T L_{n}(q)$.
Example 2.2.5. Let $n=10$ and let $t$ and $s$ be the tableaux ending in $(7,3)$ given by

$$
t=\begin{array}{|l|l|l|l|l|l|l|}
\hline 1 & 3 & 5 & 7 & 8 & 9 & 10 \\
\hline 2 & 4 & 6 & & & & \\
\hline
\end{array}
$$

Then we obtain


Remark 2.2.6. 1. It is clear that $(t, s) \mapsto \beta_{t, s}$ defines a bijection between the set $\bigcup_{\lambda \in \operatorname{Par}_{2}(n)} \operatorname{Std}(\lambda) \times \operatorname{Std}(\lambda)$ and set of arc diagrams in $T L_{n}(q)$, i.e.

$$
\left\{\beta_{t, s}, t, s \in \operatorname{Std}(\lambda), \lambda \in \operatorname{Par}_{2}(n)\right\}
$$

is a basis of $T L_{n}(q)$.
2. This basis is actually a cellular basis, see (GL96, Example 1.4), which turns $T L_{n}(q)$ into a cellular algebra. Cellular algebras were first defined by Graham and Lehrer in (GL96).

Having notation and terminology fixed, we can now describe the simple modules of $H_{n}(q)$ and $T L_{n}(q)$. However, to simplify formulas, we shall another generatorrelation presentation for the Hecke algebra. Following (Wen88, Section 2), setting

$$
\begin{equation*}
C_{i}:=T_{i}+1 \text { for } i=1, \ldots, n-1 \tag{2.11}
\end{equation*}
$$

replaces (2.1), (2.2) and (2.3) with

$$
\begin{align*}
C_{i}^{2} & =(1+q) C_{i}, & & \text { if } 1 \leq i \leq n-1,  \tag{2.12}\\
C_{i} C_{i+1} C_{i}-q C_{i} & =C_{i+1} C_{i} C_{i+1}-q C_{i+1}, & & \text { if } 1 \leq i \leq n-2,  \tag{2.13}\\
C_{i} C_{j} & =C_{j} C_{i}, & & \text { if }|i-j|>1 . \tag{2.14}
\end{align*}
$$

Remark 2.2.7. 1. The generators $C_{i}$ are more related to the generators $U_{i}$ of the Temperley-Lieb algebra than the generators $T_{i}$ are: We obtain $\phi_{1}\left(C_{i}\right)=$ $q U_{i}$ and moreover $\operatorname{ker} \phi_{1}$ is generated by

$$
C_{1} C_{2} C_{1}-q C_{i}=T_{1} T_{2} T_{1}+T_{1} T_{2}+T_{2} T_{1}+T_{1}+T_{2}+1
$$

2. One could choose to use $C_{i}^{\prime}=q-T_{i}$ instead of $C_{i}$, but then one should replace $\phi_{1}$ by $\phi_{2}$ in this and following sections.

To define $H_{n}(q)$-modules respectively $T L_{n}(q)$-modules associated to a partition $\lambda$, we still need some more notation. We follow (Wen88, Section 2):

Definition 2.2.8. For $t \in \operatorname{Std}(n)$ and $1 \leq i \leq n$ the number $d(t, i)$ is defined to be

$$
\begin{equation*}
d(t, i)=c(t, i)-r(t, i)-(c(t, i+1)-r(t, i+1)), \tag{2.15}
\end{equation*}
$$

where $c(t, i)$ denotes the column number and $r(t, i)$ the row number of $i$ in $t$.
Then we shall use the notations

$$
\begin{align*}
a_{d}^{H}(q) & =\frac{1-q^{d+1}}{1-q^{d}}  \tag{2.16}\\
a_{d}^{T L}(q) & =q^{-1} a_{d}^{H}\left(q^{2}\right)=\frac{[d+1]}{[d]} \tag{2.17}
\end{align*}
$$

supposed that $q$ (respectively $q^{2}$ ) is not a $d$ th root of unity. (2.16) and (2.17) are motivated by the corresponding coefficient under (Wen88, (2.2)). Actually $q^{2}$ not being a $d$ th root of unity is equivalent to $[d] \neq 0$.

Considering tableaux as bijections from $\langle\operatorname{Shape}(t)\rangle \rightarrow\{1, \ldots, n\}$ implies that $s_{i}(t):=s_{i} \circ t$ is the tableau obtained from $t$ by interchanging the numbers $i$ and $i+1$.

If $t \in \operatorname{Std}(n)$, such that $s_{i}(t)$ is not standard any more, then $i$ and $i+1$ must be in the same row or column, hence $d(t, i)= \pm 1$, implying

$$
\begin{equation*}
\sqrt{a_{d(t, i)}^{H}(q) a_{-d(t, i)}^{H}(q)}=0=\sqrt{a_{d(t, i)}^{T L}(q) a_{-d(t, i)}^{T L}(q)} . \tag{2.18}
\end{equation*}
$$

Following (Wen88, (2.3)), we define the $q$-analogs of Young's normal representation:

Definition 2.2.9. Let $\lambda$ be a partition of $n$.

1. Assume that $q$ is not a kth root of unity for $k=2, \ldots, n$ and let $S_{\lambda}^{H_{n}(q)}$ be the vector space with basis $\operatorname{Std}(\lambda)$. Then we define a $H_{n}(q)$-action on $S_{\lambda}^{H_{n}(q)}$ by

$$
\begin{equation*}
C_{i} \cdot t=a_{d(t, i)}^{H}(q) t+\sqrt{a_{d(t, i)}^{H}(q) a_{-d(t, i)}^{H}(q)} s_{i}(t) . \tag{2.19}
\end{equation*}
$$

2. If $[k]_{q} \neq 0$ for $k=2, \ldots, n$ and if $\lambda$ has at most two rows, we also define an action of $T L_{n}(q)$ on $S_{\lambda}^{T L_{n}(q)}$, the same vector space as above, by setting

$$
\begin{equation*}
U_{i . t}=q^{-1} C_{i} . t=a_{d(t, i)}^{T L}(q) t+\sqrt{a_{d(t, i)}^{T L}(q) a_{-d(t, i)}^{T L}(q)} s_{i}(t) \tag{2.20}
\end{equation*}
$$

(2.18) ensures that (2.19) and (2.20) are well-defined. One can check, that this defines in both cases a representation, for a proof we refer to (Wen88, Section 2). Moreover one can check that $\operatorname{ker} \phi_{1}$ from (2.7) acts by 0 , if $\lambda$ is a partition with at most 2 rows, so the representation of $H_{n}\left(q^{2}\right)$ descends to $T L_{n}(q)$ under $\phi_{1}$.

Actually these modules describe the simple ones:
Theorem 2.2.10. 1. Suppose that $q$ is not a kth root of unity for $k=2, \ldots, n$. Then $S_{\lambda}^{H_{n}(q)}$ is a simple module and the set of all $S_{\lambda}^{H_{n}(q)}, \lambda \in \operatorname{Par}(n)$ forms a complete list of inequivalent simple $H_{n}(q)$-modules.
2. If $[k] \neq 0$ for $k=2, \ldots, n$, then similarly the set of all $S_{\lambda}^{T L_{n}(q)}, \lambda \in \operatorname{Par}_{2}(n)$ forms a complete list of inequivalent simple $T L_{n}(q)$-modules.

Proof. We only sketch a proof here, a complete one can be found for example in (Wen88, Section 2). The proofs for $H_{n}(q)$ and $T L_{n}(q)$ are analoguous, thus we only consider $H_{n}(q)$. Let $M_{\lambda}$ be the $H_{n}(q)$-module defined as the vector space with basis $\operatorname{Tab}(\lambda)$ (not only standard tableaux) and the same action as above. By the above condition on $q$ not being a $k$ th root of unity for $k=2, \ldots, n$, the tableau $t \in \operatorname{Std}(n)$ satisfies $a_{d(t, i)}^{H}(q)=0$, if and only if $d(t, i)=-1$. In particular $S_{\lambda}^{H_{n}(q)}$ is an invariant subspace of $M_{\lambda}$ and therefore a submodule. The map $t \mapsto t^{\prime}$ induces an isomorphism of vector spaces between $S_{\lambda}^{H_{n}(q)}$ and $\bigoplus_{\lambda^{\prime} \subsetneq \lambda} S_{\lambda^{\prime}}^{H_{n-1}(q)}$, where $\lambda^{\prime} \subsetneq \lambda$ means, that $\lambda^{\prime}$ is a partition obtainable from $\lambda$ by removing a box. Analyzing the action of $H_{n}(q)$ actually implies, that this induces an isomorphism of $H_{n-1}(q)-$ modules. Then one can use induction to show that $S_{\lambda}$ is simple. Moreover, if $\mu$ and $\lambda$ are two inequal partitions and $n>2$, then at least one of them contains a
partition $\lambda^{\prime}$ that is not contained in the other one. In particular, $S_{\lambda}$ and $S_{\mu}$ are non-isomorphic as $H_{n-1}(q)$-modules. Moreover, if $V$ is any simple $H_{n}(q)$-module, it decomposes by induction into a sum of some $S_{\mu}^{H_{n-1}(q)}$ as a $H_{n-1}(q)$-module. So $V$ must contain a simple $S_{\lambda}^{H_{n}(q)}$, implying that $V$ is already $S_{\lambda}^{H_{n}(q)}$.

Since the branching rule $S_{\lambda} \cong \bigoplus_{\lambda^{\prime} \subseteq \lambda} S_{\lambda^{\prime}}$ is the same as for the group algebra $\mathbb{C} S_{n}$ of the symmetric group $S_{n}, H_{n}(q)$ turns out to be semisimple. A more general criterion for semisimplicity can be found in (HST15b, Theorem 5.1) for the Hecke algebra and in (HST15a, Proposition 5.1) for the Temperley-Lieb algebra.

Corollary 2.2.11. If $q$ is not a kth root of unity for $k=2, \ldots, n$, then $H_{n}(q)$ is semisimple. In particular, if $[k] \neq 0$ for $k=2, \ldots, n$, also $T L_{n}(q)$ is semisimple.

Proof. The dimension of $H_{n}(q)$ is $n$ ! by using Theorem 2.2.3. Moreover, the simple $S_{n}$-module $S_{\lambda}^{S_{n}}$ associated to the partition $\lambda$, which is the the dequantization of $S_{\lambda}^{H_{n}(q)}$, is of same dimension than $S_{\lambda}^{H_{n}(q)}$, which implies

$$
\operatorname{dim} H_{n}(q)=n!=\operatorname{dim} \mathbb{C} S_{n}=\sum_{\lambda \in \operatorname{Par}(n)}\left(\operatorname{dim} S_{\lambda}^{S_{n}}\right)^{2}=\sum_{\lambda \in \operatorname{Par}(n)}\left(\operatorname{dim} S_{\lambda}^{H_{n}(q)}\right)^{2} .
$$

Thus $H_{n}(q)$ is semisimple. $T L_{n}(q)$ is semisimple, as it is a quotient of $H_{n}\left(q^{2}\right)$.
Because we will work a lot with the branching graph of the Temperley-Lieb algebras in Figure 2, we stress this fact once again:

Corollary 2.2.12. If $[k] \neq 0$ for $k=2, \ldots, n$, then the graph $B_{2}$ introduced before (see Figure 2) is the branching graph of the algebras $T L_{1} \subset T L_{2} \subset \ldots T L_{n}$.

To avoid confusion, the inclusion $T L_{k}(q) \subset T L_{n}(q)$ is sometimes denoted by

$$
T L_{k}(q) \rightarrow T L_{n}(q), x \mapsto x \sqcup 1 .
$$

In diagrams $x \sqcup 1$ is obtained from $x$ by adding $n-k$ strands to the right of all diagrams in the expression of $x$.

Now that $T L_{n}(q)$ and $H_{n}(q)$ are semisimple with the right choice of the parameter $q$, we can actually describe $T L_{n}(q)$ by a Schur-Weyl duality statement. This only works in the semisimple case. Section 2.3 is devoted to this perspective.

Section 2.3

## A Schur-Weyl duality

If $V$ is a $r$-dimensional $\mathbb{C}$-vector space, it is in a natural way a left module of the Lie algebra $\mathfrak{g l}_{r}=M_{r} \mathbb{C}$ by left multiplication. Now by using the comultiplication of its universal envoloping algebra $U\left(\mathfrak{g l}_{r}\right)$ defined by $x \mapsto x \otimes 1+1 \otimes x$ for $x \in \mathfrak{g l}_{r}$, the tensor space $V^{\otimes n}$ becomes a left $U\left(\mathfrak{g l}_{r}\right)$-module. However, on the other side the symmetric group $S_{n}$ acts from the right on the space $V^{\otimes n}$, so does its group algebra $\mathbb{C} S_{n}$, and moreover this action commutes with the left $U\left(\mathfrak{g l}_{r}\right)$-action. Furthermore, one can actually show that their images in the endomorphism ring of $V^{\otimes n}$ are the
centralizers of each other. Applying now the powerful double centralizer theorem allows decomposing $V^{\otimes n}$ into a sum of spaces of the form $V_{\lambda} \otimes S_{\lambda}$, where $V_{\lambda}$ is a simple $U\left(\mathrm{gl}_{r}\right)$-module, i.e. a highest weight module, and where $S_{\lambda}$ is a simple $S_{n}$-module. This statement is known as the Schur-Weyl duality in the classical case.

However, this statement holds in a greater generality. We will consider here a quantized version, namely we replace $\mathbb{C} S_{n}$ by the Hecke algebra $H_{n}\left(q^{2}\right)$, the universal enveloping algebra $U\left(\mathfrak{g l}_{r}\right)$ of the Lie algebra $\mathfrak{g l}_{r}$ by the quantum group $U_{q}\left(\mathfrak{g l}_{r}\right)$ and the $\mathbb{C}$-vector space $V$ by a certain $U_{q}\left(\mathfrak{g l}_{r}\right)$-module. Since the images of $U_{q}\left(\mathfrak{g l}_{r}\right)$ and $U_{q}\left(\mathfrak{s l}_{r}\right)$ in that endomorphism ring will coincide, we will work with $\mathfrak{s l}{ }_{r}$ instead of $\mathfrak{g l}_{r}$ here. Later as a consequence in the case $r=2$, the TemperleyLieb algebra will be described as the image of the Hecke algebra $H_{n}\left(q^{2}\right)$ in the endomorphism ring.

Summarizing, this section is mainly about to the following statement:
Theorem. Assume that $q^{2}$ is not a kth root of unity for $k=2, \ldots, n$. Let $V$ be the fundamental $U_{q}\left(\mathfrak{s l}_{r}\right)$-module. Then we obtain a decomposition as $U_{q}\left(\mathfrak{s l}_{r}\right)-H_{n}\left(q^{2}\right)$ bimodules

$$
V^{\otimes n} \cong \bigoplus_{\lambda \in \operatorname{Par}_{r}(n)} S_{\lambda} \otimes V_{\lambda}
$$

where the $S_{\lambda}^{H_{n}\left(q^{2}\right)}$ are simple $H_{n}\left(q^{2}\right)$-modules, the $V_{\lambda}$ are simple $U_{q}\left(\mathfrak{s l}_{r}\right)$-modules and the sum ranges over all partitions $\lambda$ of $n$ with at most $r$ rows.

The first necessity to understand and prove the above statement is to introduce the used language. We start by recalling the definition of the centralizer.

Definition 2.3.1. For a $K$-vector space $M$ and $S \subset \operatorname{End}_{K}(M)$, the centralizer of $S$ in $\operatorname{End}_{K}(M)$ is defined to be

$$
C_{K}^{M}(S):=\left\{\phi \in \operatorname{End}_{K}(M), \phi \circ s=s \circ \phi, \forall s \in S\right\}
$$

Remark 2.3.2. If $A$ is a semisimple $\mathbb{C}$-algebra and $\rho: A \rightarrow \operatorname{End}(V)$ a morphism of $\mathbb{C}$-algebras for a complex finite dimensional vector space $V$, then $\langle A\rangle:=\rho(A)$ is again a semisimple algebra, since the property being semisimple is closed under taking submodules and quotients by semisimples.

The following version of the double centralizer theorem is not shown here, we instead refer to (Kna07, Theorem 2.43) for a version concerning simple algebras and to (KP96, Section 3.2) for a version concerning semisimple algebras:

Theorem 2.3.3 (Double Centralizer Theorem). Let $K$ be an algebraically closed field, $W$ be a finite dimensional $K$-vector space and $A \subset \operatorname{End}_{K}(W)$ semisimple.

1. Then $A^{\prime}=C_{K}^{W}(A)$, the centralizer of $A$ in $\operatorname{End}_{K}(W)$, is a semisimple subalgebra of $\operatorname{End}_{K}(W)$ and $C_{K}^{W}\left(C_{K}^{W}(A)\right)=C_{K}^{W}\left(A^{\prime}\right)=A$.
2. $W$ decomposes as an $A \otimes_{K} A^{\prime}$-module into simple $A \otimes_{K} A^{\prime}$-modules $W_{i}$ of the form $V_{i} \otimes_{K} V_{i}^{\prime}$, where
(a) $V_{i}$ is a simple $A$-module and $V_{i}^{\prime}$ a simple $A^{\prime}$-module and
(b) the $V_{i}$ form a complete set of inequivalent simple $A$-modules, so do the $V_{i}^{\prime}$ respectively. In particular, the sets of isomorphism classes of simple $A$-modules and simple $A^{\prime}$-module are in bijective correspondence.
$H_{n}\left(q^{2}\right)$ will play the role of $A^{\prime}$. The role of $A$ would normally be played by another object, namely the quantum group $U_{q}\left(\mathfrak{g l}_{r}\right)$, however four our purposes it is sufficient to consider the quantum group $U_{q}\left(\mathfrak{s l}_{r}\right)$, since the images of their actions on the later defined tensor space $V^{\otimes n}$ coincide. We briefly recall some facts:
3. The quantum group $U_{q}\left(\mathfrak{s l}_{r}\right)$ is the $\mathbb{C}$-algebra in the generators $E_{i}, F_{i}, K_{i}, K_{i}^{-1}$ for $1 \leq i \leq r-1$ subject to the usual relations, see for example (HPK91, Section 0) for the general definition of the quantum group $U_{q}(\mathfrak{g})$ associated to a complex semisimple Lie algebra $\mathfrak{g}$.
4. $U_{q}\left(\mathfrak{s l}_{r}\right)$ admits a coalgebra structure with comultiplication $\Delta$

$$
\begin{align*}
& \Delta\left(E_{i}\right)=1 \otimes E_{i}+E_{i} \otimes K_{i}, \quad \Delta\left(F_{i}\right)=K_{i}^{-1} \otimes F_{i}+F_{i} \otimes 1  \tag{2.21}\\
& \Delta\left(K_{i}\right)=K_{i} \otimes K_{i} \tag{2.22}
\end{align*}
$$

3. The set of weights is given by

$$
P=\left\{\left(\lambda_{1}, \ldots, \lambda_{r-1}\right), \lambda_{i}= \pm q^{m_{i}}, m_{i} \in \mathbb{Z}\right\} .
$$

For $\sigma \in\{ \pm 1\}^{r-1}$ we define the subset $P_{\sigma}$ of weights of type $\sigma$ by

$$
P_{\sigma}=\left\{\left(\lambda_{1}, \ldots, \lambda_{r-1}\right), \lambda_{i}=\sigma_{i} q^{m_{i}}, m_{i} \in \mathbb{Z}\right\}
$$

We restrict to weights of type $\mathbf{1}=(1, \ldots, 1)$ from now on. A weight $\lambda=$ $\left(q^{m_{1}}, \ldots, q^{m_{r-1}}\right.$ ) is called a dominant weight, if $m_{i} \geq 0$ for $i=1, \ldots, r-1$ and the set of dominant weights is denoted by $P^{\prime}$.

The next theorem can be found for example in (Ros88, Theorem 2):
Theorem 2.3.4. 1. If $q$ is not a root of unity, any finite dimensional $U_{q}\left(\mathfrak{s l}_{r}\right)$ module is semisimple.
2. The equivalence classes of simple finite dimensional $U_{q}\left(\mathfrak{s l l}_{r}\right)$-modules is indexed by the set of dominant weights.

Let $\lambda=\left(q^{m_{1}}, \ldots, q^{m_{r-1}}\right)$ be a dominant weight. If $q$ is not a $k$ th root of unity for $k=2, \ldots, \max _{i=1, \ldots, r-1}\left(m_{i}\right)$, then we can associate to $\lambda$ a partition $\mu$ of length at most $r$ by imposing the condition

$$
\mu_{i}-\mu_{i+1}=m_{i} \text { for } i=1, \ldots, r-1
$$

The partition $\mu$ is not unique, but if we restrict to $\mu \in \operatorname{Par}(n)$ for a chosen $n$, then $\mu$ is unique, if it exists.

Example 2.3.5. Let $V$ be a $n$-dimensional vector space with a chosen basis $v_{1}, \ldots$, $v_{r}$. Then we define a $U_{q}\left(\mathfrak{s l}_{r}\right)$-action on $V$ by imposing

$$
\begin{aligned}
E_{i} v_{i+1} & =v_{i}, & & E_{i} v_{j}=0, \text { if } j \neq i+1, \\
F_{i} v_{i} & =v_{i+1}, & & F_{i} v_{j}=0, \text { if } j \neq i, \\
K_{i} v_{i} & =q v_{i}, & & K_{i} v_{j}=v_{j}, \text { if } j \neq i, i+1, \\
K_{i} v_{i+1} & =q^{-1} v_{i+1} . & &
\end{aligned}
$$

Then $V$ corresponds to the simple $U_{q}\left(\mathfrak{s l}_{r}\right)$-module associated to the partition $\mu=$ $(1,0, \ldots, 0)$. This module is called the fundamental module of $U_{q}\left(\mathfrak{s l}_{r}\right)$.

The comultiplication defined in (2.21) and (2.22) induces a $U_{q}\left(\mathfrak{s l}_{r}\right)$-module structure on $V^{\otimes n}$, which is finite dimensional, thus decomposable into

$$
\begin{equation*}
V^{\otimes n}=\bigoplus_{\lambda \in P^{\prime}} m_{\lambda} V_{\lambda}^{q} \tag{2.23}
\end{equation*}
$$

as a $U_{q}\left(\mathfrak{s l}_{r}\right)$-module. Now an action of $H_{n}\left(q^{2}\right)$ on $V^{\otimes n}$ is defined as follows:
Definition 2.3.6. Let $T \in \operatorname{End}(V \otimes V)$ be the linear map defined by

$$
v_{i} \otimes v_{j} \mapsto \begin{cases}\left(q^{2}-1\right) v_{i} \otimes v_{j}-q v_{j} \otimes v_{i}, & i<j  \tag{2.24}\\ -v_{i} \otimes v_{j}, & i=j \\ -q v_{j} \otimes v_{i}, & i>j\end{cases}
$$

Then $V^{\otimes n}$ admits a right $H_{n}\left(q^{2}\right)$-action defined by

$$
\begin{equation*}
\left(v_{i_{1}} \otimes \cdots \otimes v_{i_{n}}\right) \cdot T_{i}=\left(\mathrm{id}^{\otimes i-1} \otimes T \otimes \mathrm{id}^{\otimes n-i-1}\right)\left(v_{i_{1}} \otimes \cdots \otimes v_{i_{r}}\right) \tag{2.25}
\end{equation*}
$$

where $T$ acts on the ith and $i+1$ th tensor factors.
To see that this action is well-defined, one has to check whether the relations (2.2) and (2.1) hold ((2.3) clearly does). This can be done by direct computation.

Remark 2.3.7. This action may not seem to be the usual one: In (Jim86, Section 4) an action is defined in terms of $R$-matrices using the generators $T_{i}^{\prime}=q^{-1} T_{i}$ and respective relations for the Hecke algebra. Then for example in (Du95, Section 1.2), this action is rewritten for the generators $T_{i}$ and relations (2.1)-(2.3) in a more convenient form, by defining the linear map $T$ by

$$
v_{i} \otimes v_{j} \mapsto \begin{cases}q v_{j} \otimes v_{i}, & i<j  \tag{2.26}\\ q^{2} v_{i} \otimes v_{j}, & i=j \\ \left(q^{2}-1\right) v_{i} \otimes v_{j}+q v_{j} \otimes v_{i}, & i>j\end{cases}
$$

(2.26) and (2.24) are connected by the automorphism $\beta$ on $H_{n}\left(q^{2}\right)$ defined by $T_{i} \mapsto$ $-T_{i}+q^{2}-1$, but this also intertwines the surjections $\phi_{1}$ and $\phi_{2}$ in Theorem 2.1.4, see Remark 2.1.5. Therefore choosing $\phi_{1}$ instead of $\phi_{2}$ amounts to chose (2.24) over (2.26).

For an algebra morphism $\rho: A \rightarrow \operatorname{End}_{\mathbb{C}}(V),\langle A\rangle$ denoted its image in the endomorphism ring of $V$. The following can be found in (Jim86, Proposition 3):

Proposition 2.3.8. $V^{\otimes n}$ with the right action defined in (2.24) is a $U_{q}\left(\mathfrak{s I}_{r}\right)-H_{n}\left(q^{2}\right)$ bimodule. If $H_{n}\left(q^{2}\right)$ is semisimple, i.e. if $q^{2}$ is not a kth root of unity for $k=1, \ldots, n$, then

$$
\left\langle H_{n}\left(q^{2}\right)\right\rangle=\operatorname{End}_{U_{q}\left(\mathfrak{s l}_{r}\right)}\left(V^{\otimes n}\right) \quad \text { and } \quad\left\langle U_{q}\left(\mathfrak{s l}_{r}\right)\right\rangle=\operatorname{End}_{H_{n}\left(q^{2}\right)}\left(V^{\otimes n}\right)
$$

Now Theorem 2.3.3 decomposes $V^{\otimes n}$ into simple $U_{q}\left(\mathfrak{s l}_{r}\right)-H_{n}\left(q^{2}\right)$-bimodules:
Proposition 2.3.9. If $q^{2}$ is not a kth root of unity for $k=2, \ldots, n$, then $V^{\otimes n}$ decomposes into

$$
V^{\otimes n}=\bigoplus_{\lambda \in \operatorname{Par}_{r}(n)} V_{\lambda}^{q} \otimes S_{\lambda}^{H_{n}\left(q^{2}\right)}
$$

as $U_{q}\left(\mathfrak{s l}_{r}\right)-H_{n}\left(q^{2}\right)$-bimodules, where the $S_{\lambda}$ are simple $H_{n}\left(q^{2}\right)$-modules and the $V_{\lambda}^{q}$ are simple $U_{q}\left(\mathfrak{s l}_{r}\right)$-modules.

Proof. The decomposition over $\lambda \in \operatorname{Par}(n)$ follows from Theorem 2.3.3 and Proposition 2.3.8. To see the fact, that the sum ranges only over $\lambda \in \operatorname{Par}_{r}(n)$, we argue as in (Hä99, Section 3):

1. Let $\lambda$ be a composition of $n$ of length at most $r$, i.e. a tuple $\lambda=\left(\lambda_{1}, \ldots, \lambda_{h}\right) \in$ $\mathbb{N}^{h}, h \leq r$, such that $\sum_{i} \lambda_{i}=n$, and moreover let $a$ be a row-standard tableau of shape $\lambda$, i.e. a tableau, such that its numbers along rows increase. Then we define an element $v_{a} \in V^{\otimes n}$ by setting

$$
v_{a}=v_{c(a, 1)} \otimes \cdots \otimes v_{c(a, n)},
$$

where $c(a, i)$ denotes the column number of $i$ in $a$. The elements $v_{a}$ for all such row-standard tableaux $a$ are pairwise distinct and form a basis of $V^{\otimes n}$.
2. If $\lambda$ is a composition of $n$ of length at most $r$ and if $M_{\lambda}$ denotes the subspace spanned by elements $v_{a}$, such that $a$ is a row-standard tableau of shape $\lambda$, then $M_{\lambda}$ is a $H_{n}\left(q^{2}\right)$-submodule of $V^{\otimes n}$. This can be seen by direct computation. This means in particular that $V^{\otimes n}=\bigoplus_{\lambda} M_{\lambda}$, where the sum ranges over all compositions $\lambda$ of $n$ of length at most $r$.
3. If $\lambda$ is a composition of $n$ of length at most $r$, then ordering the rows $\lambda_{i}$ lets us obtain a partition $\mu \in \operatorname{Par}_{r}(n)$ and moreover it is clear that $M_{\lambda} \cong M_{\mu}$ as right $H_{n}\left(q^{2}\right)$-modules, since the action of $H_{n}\left(q^{2}\right)$ on $v_{a}$ is determined by the column numbers, which coincide for $\mu$ and $\lambda$. In particular, we obtain

$$
\begin{equation*}
V^{\otimes n} \cong \bigoplus_{\lambda \in \operatorname{Par}_{r}(n)} n_{\lambda} M_{\lambda}, \tag{2.27}
\end{equation*}
$$

where $n_{\lambda}$ is the multiplicity of $M_{\lambda}$ in $V^{\otimes n}$.
4. For $\lambda \in \operatorname{Par}_{r}(n)$ let $M^{\lambda}=x_{\lambda} H_{n}\left(q^{2}\right)$, which equals the module $M^{\lambda}$ defined under (Mur95, Theorem 5.1) or under (DJ86, Lemma 3.2). If $a, b$ are a rowstandard tableaux of shape $\lambda$, then one can also define an element $x_{a b}$ as in (2.10), such that the module $M_{\lambda}$ is isomorphic to the module $M^{\lambda}$ by sending $v_{a}$ to $x_{t^{\lambda} a}$. This can be seen by comparing the action of $H_{n}\left(q^{2}\right)$ on $x_{t^{\lambda} a}$ in (Mur95, (4.2)) or in (DJ86, Lemma 3.2) and the action defined in (2.26). Therefore, we obtain $V^{\otimes n} \cong \bigoplus_{\lambda \in \operatorname{Par}_{r}(n)} n_{\lambda} M^{\lambda}$.
5. By (Mur95, Theorem 6.3) and (Mur95, Theorem 7.2) or by (DJ86, Corollary 4.12), $M^{\lambda}$ decomposes into simples $D^{\mu}=S_{\mu}^{H_{n}\left(q^{2}\right)}$ with $\mu \unrhd \lambda$. If $\lambda$ has at most $r$ rows, then $\mu$ has at most $r$ rows too. In particular,

$$
V^{\otimes n} \cong \bigoplus_{\lambda \in \operatorname{Par}_{r}(n)} n_{\lambda}^{\prime} S_{\lambda}^{H_{n}\left(q^{2}\right)}
$$

where $n_{\lambda}^{\prime}$ is the multiplicity of $S_{\lambda}^{H_{n}\left(q^{2}\right)}$ in $V^{\otimes n}$.
One can check in the case $r=2$, that ker $\phi_{1}$ of the map defined in (2.7) acts by zero on $V^{\otimes n}$. This leads to Proposition 2.3.10, where we refer to (LZ10, Theorem 3.5) for a proof. Actually (LZ10, Theorem 3.5) describes the situation for the generic parameter $v$ and moreover the generator-relation presentation of $H_{n}(q)$ is slightly different. However, one can also see Proposition 2.3.10 by using (Hä99, Theorem 6), which is easier to understand and also uses our generator-relation presentation of $H_{n}(q)$.

Proposition 2.3.10. If $q^{2}$ is not a kth root of unity for $k=2, \ldots, n$ and if $V$ is the fundamental $U_{q}\left(\mathfrak{s l}_{2}\right)$-module, then $\operatorname{End}_{U_{q}\left(\mathfrak{s s}_{2}\right)}\left(V^{\otimes n}\right)$ is isomorphic to the TemperleyLieb algebra $T L_{n}(q)$.

In particular, by use of Proposition 2.3.9 and Proposition 2.3 .10 we obtain a Schur-Weyl duality statement:

Corollary 2.3.11. If $q^{2}$ is not a kth root of unity for $k=2, \ldots, n$, then as $U_{q}\left(\mathfrak{s l}_{2}\right)$ $T L_{n}(q)$-bimodules, we have the decomposition

$$
V^{\otimes n}=\bigoplus_{\lambda \in \operatorname{Par}_{2}(n)} V_{\lambda}^{q} \otimes S_{\lambda}^{T L_{n}(q)}
$$

Now that we have identified the Temperley-Lieb algebra as an endomorphism ring, one could ask how the highest weight projections look like if expressed by diagramatic language. This gives rise to the theory of the (higher order) JonesWenzl projectors, which we will deal with in Section 3. However, since diagrams do not depend on $q$, we can actually also treat the non-semisimple case; this will be done in Section 3.2.

Section 3

## Idempotents in the Temperley-Lieb algebra

In this section we summarize the theory of the (higher order) Jones-Wenzl projectors, most of which can be found in (GW93). Apart from this source, we also took some inspiration of (CH15). Although these projectors correspond under SchurWeyl duality to the highest weight projectors of quantum $\mathfrak{s l}_{2}$, we define them in Section 3.1.1 by a diagramatic recursion following (CH15). The advantage of this aproach is that we can specialize "evaluable" equalities at a root of unity, where Schur-Weyl duality does not hold. Following this idea, we define a complete set of orthogonal idempotents by diagramatic relations, so-called path idempotents. Our definition in Section 3.1.2 is basically (CH15, Definition 2.17). If $S_{\lambda}$ is the simple $T L_{n}$-module corresponding to a partition $\lambda$, then it has a basis indexed by tableaux $t$ of shape $\lambda$. The path idempotent $p_{t}$ is defined in (GW93, Section 0.3 ) to be the orthogonal projection onto the one dimensional subspace spanned by the basis element $t$ in $S_{\lambda}$. With these two descriptions of the $p_{t}$ in mind, we obtain by using the diagramatic description of the path idempotents a purely diagramatic definition of the higher order Jones-Wenzl projectors in Section 3.1.3, which then correspond to the projections onto the isotopic components of the simple modules $S_{\lambda}$.

But the definitions of the (higher order) Jones-Wenzl projectors and of the path idempotents involve coefficents, which are not necessarily well-defined, if specialized at a root of unity. Therefore we need to keep track of the coefficents and moreover we need to unterstand the relations between the path idempotents to get a better understanding of the situation. Section 3.2 is devoted to this perspective, following (GW93, Section 2). Section 3.2.1 consists of a couple of technical statements, needed to construct in Section 3.2.2 a sufficient number of well-defined idempotents. Not all of them are path idempotents and they do not give rise to well-defined higher order Jones-Wenzl projectors as in the generic case, but nevertheless, we can identify with their aid the minimal central idempotents modulo the radical of the Temperley-Lieb at a root of unity.

There is a broad literature about these idempotents, the results of this section are well-known and can be found in (GW93) and (CH15), however some of the proofs in Section 3.2.1 appear to be new.

## Section 3.1

## The generic case

In Section 2, $q \in \mathbb{C}^{\times}$was a complex number, however, in this section it is helpful to step back for a moment and to work over the field $\mathbb{C}(v)$, where $v$ is a generic parameter. Moreover, we point out that the theory for $q \in \mathbb{C}^{\times}$being not a root of unity works in parallel to the generic case. To make formulas more clean it is convenient to write $T L_{n}$ for $T L_{n}(v)$ and $T L_{n, k}$ for $T L_{n, k}(v)$.

Proposition 2.3.10 states that $T L_{n}$ is isomorphic to $\operatorname{End}_{U_{v}\left(\operatorname{sL}_{2}\right)}\left(V^{\otimes n}\right)$, where $V$ is the fundamental $U_{v}\left(\mathfrak{s l}_{2}\right)$-module from Example 2.3.5. By using Corollary 2.3.11 $T L_{n}$ decomposes into

$$
\begin{align*}
T L_{n} & \cong \operatorname{End}_{U_{v}\left(s_{2}\right)}\left(V^{\otimes n}\right) \cong \operatorname{End}_{U_{v}\left(s_{2}\right)}\left(\bigoplus_{\lambda \in \operatorname{Par}_{2}(n)} V_{\lambda}^{v} \otimes S_{\lambda}\right) \cong \bigoplus_{\lambda \in \operatorname{Par}_{2}(n)} \operatorname{End}_{U_{v}\left(s_{2}\right)}\left(V_{\lambda}^{v} \otimes S_{\lambda}\right) \\
& \cong \bigoplus_{\lambda \in \operatorname{Par}_{2}(n)} \operatorname{End}_{\mathbb{C}(v)}\left(S_{\lambda}\right) \tag{3.1}
\end{align*}
$$

where $S_{\lambda}:=S_{\lambda}^{T L_{n}(v)}$ is the simple $T L_{n}(v)$-module introduced in Definition 2.2.9. Let $z_{\lambda} \in T L_{n}$ correspond to the identity in $\operatorname{End}\left(S_{\lambda}\right)$ under the above isomorphism: This is a minimal central idempotent. Therefore with (3.1), $T L_{n}$ decomposes into

$$
T L_{n}=\bigoplus_{\lambda \in \operatorname{Par}_{2}(n)} z_{\lambda} T L_{n}=\bigoplus_{\lambda \in \operatorname{Par}_{2}(n)} z_{\lambda} T L_{n} z_{\lambda}
$$

It is clear that multiplying with $z_{\lambda}$ in $T L_{n}$ now corresponds to projecting onto the $S_{\lambda}$-isotypical component of $T L_{n}$.

The Jones-Wenzl projectors are now constructed diagramatically to be these projections onto the isotypical components. Remember, that the inclusion $T L_{n} \hookrightarrow$ $T L_{n+1}$ was denoted by $x \mapsto x \sqcup 1$. First defined in (Wen87) we define as in (CH15, (2.4)) the Jones-Wenzl projectors:

Definition 3.1.1. The nth Jones-Wenzl projector $p_{n} \in T L_{n}$ is defined by the following recursive rule:

$$
\begin{array}{ll}
p_{n}=1, & \text { if } n=1, \\
p_{n}=p_{n-1} \sqcup 1-\frac{[n-1]}{[n]}\left(p_{n-1} \sqcup 1\right) h_{n-1}\left(p_{n-1} \sqcup 1\right), & \text { if } n \geq 2 . \tag{3.2}
\end{array}
$$

Although $p_{n}$ is not an arc diagram but a linear combination of these, it is convenient to illustrate $p_{n}$ by a box with $n$ incoming and outcoming strands: $p_{n}=$ $\xrightarrow[n]{n}$. Then (3.2) can be rewritten in terms of diagrams:


The following characterization will identify $p_{n}$ with $z_{\lambda}$ for $\lambda=(n)$. An equivalent version (see Remark 3.1.3) can be found in (KL94, Section 3.1).

Proposition 3.1.2. The Jones-Wenzl projectors $p_{n}$ are uniquely characterized by:

1. $p_{n}-1$ is an element of the subalgebra of $T L_{n}$ generated by $U_{1}, \ldots, U_{n-1}$ as an associative algebra.
2. $p_{n} U_{i}=0=U_{i} p_{n}$ for all $i=1, \ldots, n-1$.

Remark 3.1.3. 1. $T L_{n}$ is the unital, associative $\mathbb{C}(q)$-algebra generated by the elements $U_{1}, \ldots, U_{n-1}$. In particular, the subalgebra of $T L_{n}$ generated by $U_{1}, \ldots, U_{n-1}$ as an associative algebra does not contain the unit element $1 \in$ $T L_{n}$, though it can have another element as the unit element.
2. In presence of the second property, the first one is equivalent to $p_{n}$ being idempotent and non-zero:

- Assume that $p_{n}$ is idempotent and non-zero and satisfies the second property. If $p_{n}$ decomposes as $p_{n}=\gamma_{0}+\sum_{i} \gamma_{i} b_{i}$, where the $b_{i}$ are nonempty words in the generators $U_{1}, \ldots, U_{n-1}$ (in particular they are not equal to the unit element) and the $\gamma_{i}$ some coefficents, then idempotency and the second property implies

$$
p_{n}=p_{n}^{2}=p_{n} \gamma_{0}+\sum_{i} \gamma_{i} p_{n} b_{i}=p_{n} \gamma_{0}
$$

Since $p_{n}$ is non-zero, $\gamma_{0}$ must be one and hence $p_{n}-1$ is a linear combination of non-empty words in the generators $U_{1}, \ldots U_{n-1}$.

- If the Jones-Wenzl projectors $p_{n}$ satisfy the two properties, then they are idempotent, since $p_{n}^{2}=p_{n}\left(p_{n}-1\right)+p_{n}=0+p_{n}$.
Proof of Proposition 3.1.2. The statement is clear for $p_{1}=1$, so let $n>1$.

1. By definition $p_{n-1}$ is an element of $T L_{n-1}$, in particular, it is expressable by

$$
p_{n-1}=\gamma_{0}+\sum_{i=1}^{k} \gamma_{i} b_{i}
$$

where the $\gamma_{i} \in \mathbb{C}(v)$ are some coefficents and the $b_{i}$ are non-empty words in the generators $U_{1}, \ldots, U_{n-2}$. Since the inclusion from $T L_{n-1} \hookrightarrow T L_{n}$, given by $x \mapsto x \sqcup 1$, maps $U_{i} \rightarrow U_{i}$ for $i=1, \ldots, n-2$, also

$$
B_{n}:=\left(p_{n-1} \sqcup 1\right) U_{n-1}\left(p_{n-1} \sqcup 1\right)=\gamma_{0} U_{n-1}+\sum_{\substack{i+j=1 \\ i, j \geq 0}}^{k} \gamma_{i} \gamma_{j}\left(b_{i} \sqcup 1\right) U_{n-1}\left(b_{j} \sqcup 1\right)
$$

is a linear combination of non-empty words in the generators $U_{1}, \ldots, U_{n-1}$. In particular, $B_{n}$ is an element of the subalgebra of $T L_{n}$ generated by the elements $U_{1}, \ldots, U_{n-1}$ as an associative algebra. So is $p_{n-1}-1$ by induction hypothesis. Moreover, by using (3.2) also

$$
\begin{aligned}
p_{n}-1 & =\left(p_{n-1} \sqcup 1\right)-1-\frac{[n-1]}{[n]}\left(p_{n-1} \sqcup 1\right) U_{n-1}\left(p_{n-1} \sqcup 1\right) \\
& =\left(p_{n-1}-1\right) \sqcup 1-\frac{[n-1]}{[n]} B_{n}
\end{aligned}
$$

is an element of that subalgebra, i.e. $p_{n}$ satisfies the first property.
2. Assume by induction that the second property is true for all $U_{i}, p_{l}$ with $i \leq l$ and $l<n$. If $i<n-1$, then (3.2) yields that

$$
\begin{aligned}
p_{n} U_{i} & =\left(p_{n-1} \sqcup 1\right) U_{i}-\frac{[n-1]}{[n]}\left(p_{n-1} \sqcup 1\right) U_{n-1}\left(p_{n-1} \sqcup 1\right) U_{i} \\
& =\left(p_{n-1} \sqcup 1\right)\left(U_{i} \sqcup 1\right)-\frac{[n-1]}{[n]}\left(p_{n-1} \sqcup 1\right) U_{n-1}\left(p_{n-1} \sqcup 1\right)\left(U_{i} \sqcup 1\right) \\
& =\left(p_{n-1} U_{i} \sqcup 1\right)-\frac{[n-1]}{[n]}\left(p_{n-1} \sqcup 1\right) U_{n-1}\left(p_{n-1} U_{i} \sqcup 1\right)=0,
\end{aligned}
$$

since $U_{i}=U_{i} \sqcup 1$ under $T L_{n-1} \hookrightarrow T L_{n}$. Similarly $U_{i} p_{n}$ is 0 . What is left to show is the equation $p_{n} U_{n-1}=0=U_{n-1} p_{n}$ and by symmetry, it suffices to show that $U_{n-1} p_{n}=0$.
The first property applied to $p_{j}$ and the second to $p_{i}$ imply together

$$
\begin{equation*}
p_{i} \cdot p_{j} \sqcup 1=p_{i}\left(p_{j} \sqcup 1-1\right)+p_{i}=p_{i} \quad \forall i<n, j \leq i . \tag{3.3}
\end{equation*}
$$

Moreover, applying (3.2) and using [2] $-\frac{[n-2]}{[n-1]}=\frac{[n]}{[n-1]}$ yields

$$
\begin{equation*}
\stackrel{\mid}{\mid n-1} \tag{3.4}
\end{equation*}
$$

hence combined with (3.3), the equation

holds. In particular, $p_{n}$ also satisfies the second property.
The first and the second poperty imply, that $1-p_{n}$ is a unit element in the subalgebra generated by $U_{1}, \ldots, U_{n-1}$ as an associative algebra, in particular, $1-p_{n}$ is unique. But then also $p_{n}$ is unique.

With these characterization it is easy to identify $p_{n}$ as the element $z_{\lambda}$ where $\lambda=(n)$ is the maximal partition of $n$.

Corollary 3.1.4. $p_{n}$ equals the minimal central idempotent $z_{(n)}$.
Proof. It is sufficient to show that the element $z_{(n)}$ satisfies the two properties of Proposition 3.1.2. By Remark 3.1.3 it is actually enough to consider the second property, since $z_{(n)}$ is non-zero and idempotent. Now $S_{(n)}$ is the one dimensional
$T L_{n}$-module spanned by the tableau $t^{(n)}$, which implies that $d=d\left(t^{(n)}, i\right)=-1$ for all $i$, hence $a_{d}=0$, where $a_{d}$ is defined in (2.17) and $d\left(t^{(n)}, i\right)$ in (2.15). In particular, (2.20) implies that all generators $U_{i}$ act by 0 on $S_{(n)}$. Since multiplying with $z_{(n)}$ corresponds to projecting onto the $S_{(n)}$-isotypical component and since $z_{(n)}$ is central, this already implies the second property.

Following ( CH 15 , Section 2.14), the next step will be to express path idempotents in diagramatic language. They are the main ingredient to define analogs of the Jones-Wenzl projectors corresponding to $\lambda \neq(n)$, which will be done in Section 3.1.3, and moreover they are also the main subject of Section 3.2.1.

## Section 3.1.2

## Path idempotents

Now that the projection onto $S_{(n)}$ is expressed in terms of diagrams, a natural idea would be to express the other projections onto the $S_{\lambda}$-isotypical parts in diagrams as well. However, the case for $\lambda=(n)$ is "easier" as for general partitions $\lambda \in$ $\operatorname{Par}_{2}(n)$, already the basis of $S_{(n)}$ consists only of one element, namely the only standard tableau of $(n)$. Therefore to define higher order Jones-Wenzl projectors, which will be postponed to Section 3.1.3, it would be a good idea to define first an analog of the orthogonal projection $S_{\lambda} \rightarrow \mathbb{C} t$ in terms of diagrams, where $t \in$ $\operatorname{Std}(\lambda)$ is a basis element of $S_{\lambda}$. This is exactly the outline of this section. Following (CH15, Definition 2.17), a definition of elements $p_{t}$ is given in the beginning of this section, then a couple of properties are proven, to show in the end of this section, that $p_{t}$ actually corresponds to the orthogonal projection $S_{\lambda} \rightarrow \mathbb{C} t$.

A standard tableau $t$ in $\operatorname{Std}(\lambda)$ will always be identified with its path in the branching graph $B_{2}$. The path $t$ has always an extension $t^{+}$of shape $\left(\lambda_{1}+1, \lambda_{2}\right)$ and if $\lambda_{1}>\lambda_{2}$, it also has the extension $t^{-}$of shape $\left(\lambda_{1}, \lambda_{2}+1\right)$. Moreover $t^{\prime}$ was the subpath of $t$ of length $n-1$.

First the coefficients of the later defined path idempotents are defined:
Definition 3.1.5. Let $t \in \operatorname{Std}(n)$ be a standard tableau. The coefficent $f_{t} \in \mathbb{C}(v)$ is defined by the following recursive rule:

- If $n=1$, then the coefficient $f_{t}$ for the unique $t$ in $\operatorname{Std}(1)$ is defined to be 1 .
- If $n \geq 2, f_{t}$ is said to be

$$
f_{t}= \begin{cases}f_{t^{\prime}}, & \text { if } t=t^{\prime+}, \\ \frac{[k]}{[k+1]} f_{t^{\prime}}, & \text { if } t=t^{\prime-},\end{cases}
$$

where $k=\lambda_{1}-\lambda_{2}$ and $\lambda=\operatorname{Shape}\left(t^{\prime}\right)$.

## Remark 3.1.6.

1. $T L_{n}$ was only defined for $n \geq 1$. Though it seems a bit strange, for the sake of the next definition, it is convenient to formally set $T L_{0}:=\mathbb{C}(v)$ and to identify $T L_{0}$ with the unital algebra generated by the "empty" diagram.

In this setting the 0th Jones-Wenzl projector $p_{0}$ is defined to be the empty diagram with coefficent 1 .
2. Similar as $p_{n}$ was illustrated by $\xrightarrow[n]{n}$ and because the diagramatic arguments will get more complicated in the future sections, also an element $x \in T L_{n, k}$, will be illustrated by a grey box $x=, \quad \begin{gathered}i \\ \vdots\end{gathered}$, with $n$ incoming and $k$ outcoming strands.

The map $T L_{n, k} \rightarrow T L_{k, n}$ induced by flipping diagrams vertically was denoted by $x \mapsto \tilde{x}$. With this notation and the above remark in mind, it is possible to define elements $p_{t} \in T L_{n}$ indexed by paths $t \in \operatorname{Std}(\lambda)$ :

Definition 3.1.7. Let $t \in \operatorname{Std}(n)$ be a path. The element $p_{t} \in T L_{n}$ is defined by the following rule:

- For $n=1$ and $t \in \operatorname{Std}(1), p_{t}$ is defined to be $1 \in T L_{1}$.
- If $n \geq 2$, assume that $\operatorname{Shape}\left(t^{\prime}\right)=\lambda$ with $k=\lambda_{1}-\lambda_{2}$ and that $p_{t^{\prime}}$ is defined for $t^{\prime}$ and satisfies $p_{t^{\prime}}=f_{t^{\prime}} \cdot x p_{k} \tilde{x}=f_{t^{\prime}} \cdot \frac{x_{1}}{\substack{x \\ x}}$, , where $x \in T L_{n-1, k}$ and $f_{t}$ is defined in Definition 3.1.5. Then we define $p_{t}$ by

Remark 3.1.8. 1. These elements $p_{t}$ will turn out to be the wanted path idempotents. However, it is not known yet that they are idempotent, likewise till idempotency is proven, the name path idempotent shall not be used.
2. Substituting (3.2) into (3.5) yields the reccurence

$$
\begin{equation*}
p_{t} \sqcup 1=p_{t^{+}}+p_{t^{-}} \tag{3.6}
\end{equation*}
$$

3. If a simple transposition $s_{i}$ is not admissible for a path $r$, i.e. if the tableau $s_{i}(r)$ is not standard, it is not hard to check that

$$
\begin{equation*}
U_{i} p_{r}=0=p_{r} U_{i} \tag{3.7}
\end{equation*}
$$

which is left as an exercise to the reader.
Example 3.1.9. 1. Let $\lambda=(n)$ and $t$ be its unique standard tableau. Then $p_{t}$ equals to $\stackrel{n}{\square \quad 1}=p_{n}$.
2. Let $\lambda=(n-1,1)$ and let $t$ be the standard tableau with entry $i \in\{1, \ldots, n\}$ in the second row. The corresponding path and its path idempotent are illustrated in Figure 4.


Figure 4: A path $t$ of shape $(n-1,1)$ and the element $p_{t}$.
Idempotency of the element $p_{t}$ can directly be shown by using its definition:
Lemma 3.1.10. The element $p_{t}$ is idempotent, i.e. $p_{t}^{2}=p_{t}$.
Proof. Let by Definition 3.1.7 the element $p_{t}$ be given by $f_{t} \cdot x p_{k} \tilde{x} \in T L_{n}$. We first show the following equation by using induction over $n$ :

$$
\begin{equation*}
p_{k} \tilde{x} x p_{k}=\frac{1}{f_{t}} p_{k} \tag{3.8}
\end{equation*}
$$

Proof of the equation (3.8). For $t \in \operatorname{Std}(1)$ (3.8) is clearly true. Now assume for $p_{t^{\prime}}=f_{t^{\prime}} \cdot y p_{k} \tilde{y} \in T L_{n-1}$ that (3.8) holds, i.e. assume that

$$
\begin{equation*}
p_{k} \tilde{y} y p_{k}=\frac{1}{f_{t^{\prime}}} p_{k} \tag{3.9}
\end{equation*}
$$

There are the following two cases:

1. If $t=t^{\prime+}$, then $f_{f}=f_{t^{\prime}}$ and by Definition 3.1.7 $p_{t}$ is of the form

$$
p_{t}=f_{t} \cdot(y \sqcup 1) p_{k+1}(\tilde{y} \sqcup 1)=f_{t^{\prime}}(y \sqcup 1) p_{k+1}(\tilde{y} \sqcup 1) .
$$

This implies with $p_{k+1}=p_{k+1}\left(p_{k} \sqcup 1\right)$ (see (3.3)) and with (3.9), that

$$
\begin{aligned}
p_{k+1}(y \sqcup 1)(\tilde{y} \sqcup 1) p_{k+1} & =p_{k+1}\left(p_{k} \sqcup 1\right)(y \sqcup 1)(\tilde{y} \sqcup 1)\left(p_{k} \sqcup 1\right) p_{k+1} \\
& \left.=p_{k+1}\left(\left(p_{k} y \tilde{y} p_{k}\right) \sqcup 1\right)\right) p_{k+1} \\
& =\frac{1}{f_{t^{\prime}}} p_{k+1}\left(p_{k} \sqcup 1\right) p_{k+1}=\frac{1}{f_{t}} p_{k+1},
\end{aligned}
$$

thus (3.8) holds for $p_{t}$ in this case.
2. Now assume that $t=t^{\prime-}$ and let $\lambda=\operatorname{Shape}\left(t^{\prime}\right)$ with $k=\lambda_{1}-\lambda_{2}$. By Definition 3.1.7 $p_{t}$ is of the form


So, (3.9) , (3.4) and idempotency of $p_{k+1}$ together imply that


But this is just (3.8) for $p_{t}$.
Since $p_{k}$ is idempotent, (3.8) implies that

$$
p_{t} p_{t}=f_{t}^{2} x p_{k} \tilde{x} x p_{k} \tilde{x}=f_{t} x p_{k} \tilde{x}=p_{t}
$$

Now that the element $p_{t}$ is idempotent, we will call it the path idempotent associated to $t$.

Remark 3.1.11. We saw in the proof of Lemma 3.1.10, that $p_{k} \tilde{x} x p_{k}=\frac{1}{f_{t}} p_{k}$ for a path idempotent $p_{t}=f_{t} \cdot x p_{k} \tilde{x}$. In particular, this implies, that the coefficent of 1 in $\tilde{x} x$ is exactly $1 / f_{t}$.

The following termininology, following (CH15, Definition 2.6), may seem artificial, but actually just formalizes an easy idea. $\sum T L_{n, k} T L_{k} T L_{k, d} \subset T L_{n, d}$ is meant to be the submodule in $T L_{n, d}$ generated by the set $T L_{n, k} T L_{k} T L_{n, d} \subset T L_{n, d}$.

Definition 3.1.12. We say an element $a \in T L_{n}$ has through-degree $k$, if $a \in$ $\sum T L_{n, k} T L_{k} T L_{k, n} \subset T L_{n}$ for $k \leq n$ minimal. Similar, we say that $a \in T L_{n, l}$ has through-degree $k$, if $a \in \sum T L_{n, k} T L_{k} T L_{k, l} \subset T L_{n, l \text {. }}$ for $k \leq n$ minimal.

Example 3.1.13. An arc diagram $a \in T L_{n}$ has through-degree $k$ if and only if it has exactly $k$ vertical lines, where a line is vertical if and only if its endpoints are not on the same edge. If $a=\sum_{b} \gamma_{b} b$ is a linear combination of arc diagrams $b$ with $\gamma_{b} \in \mathbb{C}(v)$, then $a$ has through-degree $k$ if and only if $k$ is the maximal throughdegree of the summands $b$ of $a$. In particular,

- the generator $U_{i} \in T L_{n}$ has through-degree $n-2$,
- $p_{n}$ has through-degree $n$, since its coefficent of 1 is 1 ,
- $x$ in $p_{t}=f_{t} \cdot x p_{k} \tilde{x}$ can assumed to have through-degree $k$, since $p_{k}$ annihilates $U_{i}$ (see Proposition 3.1.2) and
- the path idempotent $p_{t}$ has also through-degree $k$, since $p_{k} \in T L_{k}$ has throughdegree $k$.

The next step is to show orthogonality of the path idemptotents $p_{t}, t \in \operatorname{Std}(n)$. However, it is easier to treat first the following special case:

Lemma 3.1.14. Let $s, t \in \operatorname{Std}(n)$ be two paths of different shape. Then $p_{t} p_{s}=0$.
Proof. By Definition 3.1.7, $p_{t} p_{s}$ is of the form

$$
\begin{equation*}
p_{t} p_{s}=f_{t} f_{s} \cdot x p_{k} \tilde{x} y p_{d} \tilde{y} \tag{3.10}
\end{equation*}
$$

and without loss of generality, $k$ is smaller than $d$. The element $x$ is expressable as $x=\sum_{i} x_{i} x_{i}^{\prime}$, where $x_{i} \in T L_{n, d}$ and $x_{i}^{\prime} \in T L_{d, k}$ and moreover each $x_{i}^{\prime} p_{k} \tilde{x} y$ decomposes as

$$
x_{i}^{\prime} p_{k} \tilde{x} y=\sum_{i} \gamma_{i} b_{i} \text {, where } \gamma_{i} \in \mathbb{C}(v) \text { and } b_{i} \in T L_{d} .
$$

Since $p_{k} \in T L_{k}$ and $x_{i}^{\prime} \in T L_{d, k}$, it is clear that every summand $b_{i}$ is of throughdegree at most $k$. But $k<d$ implies then that, every $b_{i}$ is a linear combination of non-empty words in the generators $U_{1}, \ldots, U_{d-1}$, i.e. the coefficent of $1 \in T L_{d}$ in $b_{i}$ is 0 . By the second property in Proposition 3.1.2, this implies that $b_{i} p_{l}=0$, and hence $x_{i}^{\prime} p_{k} \tilde{x} y p_{l}=0$. Substituting this into (3.10) results in

$$
p_{t} p_{s}=f_{t} f_{s} \cdot\left(\sum_{i} x_{i} x_{i}^{\prime} p_{k} \tilde{x} y p_{l}\right) \tilde{y}=0 .
$$

Orthogonality of $p_{s}$ and $p_{t}$ for $\operatorname{Shape}(t) \neq \operatorname{Shape}(s)$ and idemptotency have the following important consequence:

Lemma 3.1.15. Let $T$ be an extension of $t \in \operatorname{Std}(n)$. Then $p_{t} p_{T}=p_{T}$.
Proof. Let $T \in \operatorname{Std}(n+k)$ for $k \geq 1$. We proceed by induction over $k$.

1. If $k=1$, then let $p_{\tau}$ correspond to the other extension $\tau$ of $t$ (if $\tau$ does not exist, set $p_{\tau}=0$ ). Then the equation $p_{\tau} p_{T}=0$ holds by Lemma 3.1.14, since $\tau$ and $T$ are of different shape (if $\tau$ does not exist, $p_{\tau} p_{T}=0$ holds anyway). In particular, it follows by using (3.6) and Lemma 3.1.10, that

$$
p_{t} p_{T}=\left(p_{\tau}+p_{T}\right) p_{T}=p_{T}^{2}=p_{T} .
$$

2. If $k \geq 2$, then induction hypothesis and the above case together yield

$$
p_{t} P_{T}=\left(p_{t} p_{T^{\prime}}\right) P_{T}=p_{T^{\prime}} p_{T}=p_{T} .
$$

Now general orthogonality follows easily as a consequence of the previous two statements:

Corollary 3.1.16. The elements $p_{t}, t \in \operatorname{Std}(n)$ are pairwise orthogonal.
Proof. Let $t, s \in \operatorname{Std}(n)$ be two different paths. If they are of different shape, then $p_{t} p_{s}=0$ holds by Lemma 3.1.14. On the other hand, if they are of same shape, there exist subpaths $\tau$ of $t$ and $\sigma$ of $s$ of different shape, since $t$ and $s$ are different. Therefore, using Lemma 3.1.15 and Lemma 3.1.14 gives

$$
p_{t} p_{s}=p_{t} p_{\tau} p_{\sigma} p_{s}=p_{t} \cdot 0 \cdot p_{s}=0
$$

Now that orthogonality is known, completeness of the path idempotents is still missing. The following proposition summarizes the situation, see also ( CH 15 , Proposition 2.19) and (CH15, Theorem 2.20):

Proposition 3.1.17. The elements $p_{t}, t \in \operatorname{Std}(n)$ are pairwise orthogonal idempotents and sum up to the identity $1_{n} \in T L_{n}$, i.e. they form a complete set of orthogonal idempotents.

Proof. The only thing to show is that $1=\sum_{s} p_{s}$. since by Lemma 3.1.10 $p_{t}$ is idempotent and by Corollary 3.1.16 the elements $p_{t}, t \in \operatorname{Std}(n)$ are pairwise orthogonal. To do so we proceed by induction over $n$. The case $n=1$ is clear, since there is only one path $s \in \operatorname{Std}(1)$, which equals $p_{1}=1$. If $n \geq 2$, induction hypothesis gives the decomposition

$$
T L_{n-1} \ni 1_{n-1}=\sum_{r \in \operatorname{Std}(n-1)} p_{r}
$$

If $r(t, n)$ denotes the row index of $n$ in $t$, then (3.6) implies

$$
T L_{n} \ni 1_{n}=1_{n-1} \sqcup 1=\sum_{r \in \operatorname{Std}(n-1)} p_{r} \sqcup 1=\sum_{r \in \operatorname{Std}(n-1)}\left(p_{r^{+}}+p_{r^{-}}\right)=\sum_{\substack{t \in \operatorname{Std}(n) \\ r(t, n)=1}} p_{t}+\sum_{\substack{t \in \operatorname{Std}(n) \\ r(t, n)=2}} p_{t}
$$

But since since $r(t, n)$ is either 1 or 2 for all $t \in \operatorname{Std}(n)$, this already means that

$$
1_{n}=\sum_{t \in \operatorname{Std}(n)} p_{t}
$$

Now that the elements $p_{t}$ form a complete set of orthogonal idempotents, there are a few technical lemmas left to prove in this section. We start with the following property, which is shown by only using the definition of the path idempotents:

Lemma 3.1.18. Let $s, t \in \operatorname{Std}(\lambda)$ with $\lambda \in \operatorname{Par}_{2}(n)$, such that $s_{i}(s)=t$ for some $i$. Then the following equation holds:

$$
\begin{equation*}
U_{i}\left(p_{t}+p_{s}\right)=\left(p_{t}+p_{s}\right) U_{i} \tag{3.11}
\end{equation*}
$$



Figure 5: A path $t$ and $s=s_{n-1}(t)$.

Proof. Assume that $i=n-1$ and let $r=t^{\prime \prime}=s^{\prime \prime}$ be the maximal common subpath of $t$ and $s$ of shape $\mu=\lambda^{(n-2)}$. The situation is illustrated in Figure 5. With $d=\mu_{1}-\mu_{2}=\lambda_{1}-\lambda_{2}$, Definition 3.1.7 states that $p_{r}, p_{t}$ and $p_{s}$ are given by

$p_{t}=f_{r} \frac{[d+1]}{[d+2]}$
 and $p_{s}=f_{r} \frac{[d]}{[d+1]}$


Applying the recursive formula (3.2) for $p_{d+1}$ yields


Now adding $p_{s}$ to (3.12) and simplifying imply that $\frac{1}{f_{r}}\left(p_{t}+p_{s}\right)$ equals to

Therefore mutliplying $\frac{1}{f_{r}}\left(p_{t}+p_{s}\right)$ from the left with $U_{n-1}=| | \bigvee$ gives


It is clear that (3.13) must be vertically symmetric, so also $\frac{1}{f_{r}}\left(p_{s}+p_{t}\right) U_{n-1}$ is of this form. Now this argument generalizes to $1 \leq i<n-1$, since Definition 3.1.7 only uses the dashed Jones-Wenzl projector ${ }_{-1, L_{-1}}^{\llcorner }$

Orthogonality and (3.7) have the following consequence:
Corollary 3.1.19. Let $t, s \in \operatorname{Std}(n)$ be two different paths. Assume that $s_{i}$ is admissible for $s$ and that $t \neq s_{i}(s)$. Then $p_{t} U_{i} p_{s}=0$ holds.

If $s_{i}$ is not admissible for $s$, then the statement is also true by (3.7).
Proof. Let $w=s_{i}(s)$. Then Lemma 3.1.18 and Corollary 3.1.16 imply together that

$$
p_{t} U_{i} p_{s}=p_{t} U_{i}\left(p_{s}+p_{w}\right) p_{s}=p_{t}\left(p_{s}+p_{w}\right) U_{i} p_{s}=0
$$

A path $t \in \operatorname{Std}(\lambda)$ can also be identified with a sequence of signs $\epsilon_{1}(t), \ldots, \epsilon_{n}(t)$ in $\{ \pm 1\}$, where $\epsilon_{i}(t)=1$, if $i$ is in the first row and $\epsilon_{i}(t)=-1$, if $i$ is in the second row of $t$. Alternatively $\epsilon_{i}(t)$ encodes the $i$ th step of $t$ seen as a path in the branching graph (Figure 2), where -1 corresponds to a step to the left and +1 to one to the right.

To show that the path idempotents correspond to the projections $S_{\lambda} \rightarrow \mathbb{C} t$ and also for Section 3.2.1, another technical result is needed, which can also be found in (GW93, Section 0). For readability we will write $p_{t^{\prime}}$ instead of $p_{t^{\prime}} \sqcup 1$.

Lemma 3.1.20. Let $t \in \operatorname{Std}(\mu)$ and $\mu \in \operatorname{Par}_{2}(n-1)$, such that $t^{-}$exists and let $k=\mu_{1}-\mu_{2}$.

1. If $\epsilon_{n-1}(t)=1$, then the following equations hold:

$$
p_{t^{-}}=\frac{[k]}{[k+1]} p_{t} U_{n-1} p_{t} \text { and } p_{t^{+}}=p_{t}-\frac{[k]}{[k+1]} p_{t} U_{n-1} p_{t}
$$

2. If $\epsilon_{n-1}(t)=-1$, then we obtain

$$
p_{t^{-}}=p_{t}-\frac{[k+2]}{[k+1]} p_{t} U_{n-1} p_{t} \text { and } p_{t^{+}}=\frac{[k+2]}{[k+1]} p_{t} U_{n-1} p_{t} .
$$

Proof. 1. If $\epsilon_{n-1}(t)=+1$, then we are in the first situation of Figure 6. By


Figure 6: $\epsilon_{n-1}(t)=+1$ on the left and $\epsilon_{n-1}(t)=-1$ on the right hand side.
Definition 3.1.7, $p_{t^{\prime}}$ is of the form $p_{t^{\prime}}=f_{t^{\prime}} x p_{k-1} \tilde{x}$ and consequently $p_{t^{\prime}}, p_{t}$ and $p_{t^{-}}$are given by

Now $p_{t} U_{n-1} p_{t}$ looks like

since $p_{k}\left(p_{k-1} \sqcup 1\right)=p_{k}$ and since the subdiagram $p_{k-1} x \tilde{x} p_{k-1}$ collapses to $f_{t^{\prime}}^{-1} p_{k-1}$ by Remark 3.1.11. But this already implies the identity for $p_{t^{-}}$. On the other hand, the identity for $p_{t^{+}}$follows from (3.6).
2. If we had that $\epsilon_{n-1}(t)=-1$, then we were in the second situation of Figure 6. Similar as in the previous case, we know that $p_{t^{\prime}}=f_{t^{\prime}} x p_{k+1} \tilde{x}$, which implies that $p_{t^{\prime}}, p_{t}$ and $p_{t^{+}}$are of the form


Considering $p_{t} U_{n-1} p_{t}$ results in

which implies the equation for $p_{t^{+}}$. As before, the identity for $p_{t^{-}}$follows from (3.6).

We end this section as anounced before by showing the following proposition:
Proposition 3.1.21. Let $t \in \operatorname{Std}(\lambda)$. Then the path idempotent $p_{t}$ acts on $S_{\lambda}$ by fixing $t$ and sending every other basis element $s \neq t$ of $S_{\lambda}$ to 0 . Moreover $p_{t}$ acts by 0 on $S_{\mu}$ if $\mu \neq \lambda$.

Proof. Let $\lambda \in \operatorname{Par}_{2}(n)$ be a partition. We prove the statement by using induction over $n$. For $n=1$ this is clear, since there is only one partition, one path and one path idempotent. Now assume that $n \geq 2$ and let $t \in \operatorname{Std}(\lambda)$ be a path. Moreover, let $\mu$ be the shape of $t^{\prime}$ and set $k=\mu_{1}-\mu_{2}$. If $0<\lambda_{2}<\lambda_{1}$, there exists another partition $v$ such that

$$
S_{\lambda} \cong S_{\mu} \oplus S_{v}
$$

as $T L_{n-1}$-modules. If otherwise $0=\lambda_{2}$ or $\lambda_{2}=\lambda_{1}$, then $S_{\lambda}$ is isomorphic to $S_{\mu}$ as $T L_{n-1}$-modules. In both cases, $p_{t^{\prime}}$ acts on a path $s$ as it acts on $s^{\prime}$, which means explicitly that

$$
\begin{equation*}
p_{t^{\prime}}, t=t \tag{3.14}
\end{equation*}
$$

and furthermore, if $s_{n-1}$ is admissible for $t$, this also means that

$$
\begin{equation*}
p_{t^{\prime} \cdot s_{n-1}(t)}=0 . \tag{3.15}
\end{equation*}
$$

The first step is to show that also $p_{t}$ fixes $t$. There are the following cases:

1. Assume that $\epsilon_{n-1}(t)=1$ and $\epsilon_{n}(t)=-1$. Lemma 3.1.20 gives a description for $p_{t}$, namely

$$
\begin{equation*}
p_{t}=\frac{[k]}{[k+1]} p_{t^{\prime}} U_{n-1} p_{t^{\prime}} . \tag{3.16}
\end{equation*}
$$

Therefore applying (3.16), (2.20), (3.14) and (3.15) all together imply that

$$
\begin{align*}
p_{t} . t & =\frac{[k]}{[k+1]} p_{t^{\prime}} U_{n-1} p_{t^{\prime}} . t=\frac{[k]}{[k+1]} p_{t^{\prime}} U_{n-1} \cdot t \\
& =\frac{[k]}{[k+1]} p_{t^{\prime}}\left(a_{d(t, n-1)}^{T L}(v) t+\sqrt{a_{d(t, n-1)}^{T L}(v) a_{-d(t, n-1)}^{T L}(v)} s_{n-1}(t)\right) \\
& =\frac{[k]}{[k+1]} a_{d(t, n-1)}^{T L}(v) t . \tag{3.17}
\end{align*}
$$

The condition $\epsilon_{n-1}(t)=1$ encodes that $n-1$ is in the first row of $t$ and on the other hand $\epsilon_{n}(t)=1$ means that $n$ is in the second row. This means that

$$
\begin{aligned}
d(t, n-1) & =c(t, n-1)-r(t, n-1)-(c(t, n)-r(t, n))=\lambda_{1}-1-\left(\lambda_{2}-2\right) \\
& =\lambda_{1}-\lambda_{2}+1=\mu_{1}-\left(\mu_{2}+1\right)+1=k
\end{aligned}
$$

which implies with (2.17) that $a_{d(t, n-1)}^{T L}(v)=\frac{[k+1]}{[k]}$. Substituting this into (3.17) shows that $p_{t} . t=t$.
2. Now assume that $\epsilon_{n-1}(t)=1=\epsilon_{n}(t)$. This means $d(t, n-1)$ equals -1 and moreover it means that $a_{d(t, n-1)}^{T L}(v)=0$, hence $U_{n-1}$ acts by 0 on $t$. Lemma 3.1.20 describes $p_{t}$ by

$$
p_{t}=p_{t^{\prime}}-\frac{[k]}{[k+1]} p_{t^{\prime}} U_{n-1} p_{t^{\prime}},
$$

implying

$$
p_{t} . t=p_{t^{\prime} \cdot t}-\frac{[k]}{[k+1]} p_{t^{\prime}} U_{n-1} p_{t^{\prime} . t}=t-\frac{[k]}{[k+1]} p_{t^{\prime}} U_{n-1} . t=t .
$$

3. The cases, where $\epsilon_{n-1}(t)=-1$ and $\epsilon_{n}(t)= \pm 1$, work out in a similar way; they are left as an exercise to the reader.

Now that $p_{t}$ is known to fix the basis element $t \in S_{\lambda}$, let $s \in S_{\mu}$ for $\mu \in \operatorname{Par}_{2}(n)$ be another basis element different from $t$. Since $p_{s}$ fixes $s$ and since $p_{s}$ and $p_{t}$ are orthogonal, it follows easily that

$$
p_{t} \cdot s=p_{t} p_{s . s}=0 . s=0 .
$$

In particular, the statement is shown.
Having the path idempotents $p_{t}$ properly introduced, it is time to turn the focus to the so-called higher order Jones-Wenzl projectors. The next section is devoted to define these and to characterize them by unique properties, as we did for the Jones-Wenzl projector in Proposition 3.1.2.

Following (CH15, Definition 2.23), we define the higher order Jones-Wenzl projectors corresponding to a partition as the sum of its path idempotents:

Definition 3.1.22. The kth higher order Jones-Wenzl projector is given by $p_{n, k}=$ $\sum_{t \in \operatorname{Std}(\lambda)} p_{t}$, where $\lambda=\left(\lambda_{1}, \lambda_{2}\right) \in \operatorname{Par}_{2}(n)$ is the unique diagram with $\lambda_{1}-\lambda_{2}=k$.

The first goal is to give a nice characterization of the $p_{n, k}$, which can be found in (CH15, Theorem 2.26). These properties are very similar to those of the JonesWenzl projectors in Proposition 3.1.2 and proven mutatis mutandis.
Theorem 3.1.23. The higher order Jones-Wenzl projectors $p_{n, k}$ in the algebra $T L_{n}$ are uniquely characterized by the following properties:

1. The element $p_{n, k} \in T L_{n}$ has through-degree $k$.
2. For any d and $a \in T L_{d, n}$ of through-degree $j<k$ the equalities ap$n, k=0$ and $p_{n, k} \tilde{a}=0$ hold.
3. If $a \in T L_{d, n}$ is of through-degree $k$, then we obtain that $a p_{n, k}=a+b$, where $b \in T L_{d, n}$ is an element of through-degree $j<k$.

Remark 3.1.24. Elements $q_{n, k}$ satisfying the above three properties are always idempotent: The third property and the first imply

$$
q_{n, k}^{2}=q_{n, k}+b,
$$

where $b$ is of through-degree $j<k$. Multiplying with $q_{n, j}$ from the right and the second property let us deduce

$$
\begin{equation*}
0=q_{n, k}^{2} q_{n, j}=q_{n, k} q_{n, j}+b q_{n, j}=b q_{n, j}=b+c, \tag{3.18}
\end{equation*}
$$

where $c$ is of through-degree $i<j$. In particular, $b$ must be 0 and hence $q_{n, k}^{2}=q_{n, k}$. Proof of Theorem 3.1.23. It is easy to see that $p_{n, k}$ satisfies the three properties:

1. The first property follows by definition, since $p_{n, k}=\sum_{t \in \operatorname{Std}(\lambda)} p_{t}$ and by Example 3.1.13, $p_{t}$ has through-degree $k$.
2. The second property follows from the fact that $U_{i} p_{k}=0=p_{k} U_{i}$ for all $U_{i} \in T L_{k}$, see Proposition 3.1.2.
3. The third follows from $a=a 1_{n}=\sum_{j=1}^{n} a p_{n, j}=\sum_{j=1}^{k} a p_{n, j}$, where we used Proposition 3.1.17.

Suppose that some element $e$ satisfies the three properties. If $j<k$, then the second property for $e$ implies that $p_{n, j} e=0$. If $j>k$ then the second property for $p_{n, j}$ implies $p_{n, j} e=0$. Now both together imply the equation

$$
\begin{equation*}
e=e 1_{n}=\sum_{j=1}^{n} e p_{n, j}=e p_{n, k} . \tag{3.19}
\end{equation*}
$$

On the other hand, the third property for $e$ implies $e p_{n, k}=p_{n, k}+b$, where $b$ is of through-degree $j<k$. This and (3.19) gives

$$
e=e p_{n, k}=e p_{n, k}^{2}=\left(p_{n, k}+b\right) p_{n, k}=p_{n, k}^{2}=p_{n, k}
$$

Corollary 3.1.25. The idempotent $p_{n, k}$ corresponds to the minimal central idempotent $z_{\lambda}$, where $\lambda_{1}-\lambda_{2}=k$. In particular, $p_{n, k}$ is central.

Proof. Proposition 3.1.21 states, that the path idempotent $p_{t}$ for $t \in \operatorname{Std}(\lambda)$ acts on $S_{\lambda}$ by fixing $t$ and sending $s \neq t$ to 0 . This directly implies, that $p_{n, k}=\sum_{t \in \operatorname{Std}(\lambda)} p_{t}$ acts as the identity on $S_{\lambda}$. Since the identity $z_{\lambda}$ in $\operatorname{End}\left(S_{\lambda}\right)$ is unique, we obtain $z_{\lambda}=p_{n, k}$.

Corollary 3.1.26. If $\operatorname{Shape}(t) \neq \operatorname{Shape}(s)$, then $p_{t} T L_{n} p_{s}=0$ is trivial.
Proof. Let $\lambda=\operatorname{Shape}(t)$ and $d=\lambda_{1}-\lambda_{2}$. Then Proposition 3.1.17 and Corollary 3.1.25 imply

$$
p_{t} T L_{n} p_{s}=p_{t} p_{d, n} T L_{n, d} p_{s}=p_{t} T L_{n, d} p_{d, n} p_{s}=0
$$

The chosen aproach of this section makes it possible to get more insight concerning the involved coefficents. This is carried out in detail in the next section. We stress that the next section does only use diagramatic arguments, however it tells the same story as (GW93).

## Section $3.2 \quad$ Specialization at a root of unity

In this section, we let $q \in \mathbb{C}^{\times}$denote a fixed primitive $2 l$ th root of unity, where $l$ is at least 3 , i.e. we assume that $q \neq \pm i$. We need this in this section, since otherwise $[2]=0$, so nearly all proofs of this section were not valid.

Now $T L_{n}(q)$ is not semisimple anymore, compare (HST15a, Proposition 5.1), so not all Jones-Wenzl projectors are present. Already some path idempotents cause problems, since their construction involves Jones-Wenzl projectors and quantum integers $[k]$, which are not always well-defined at $v=q$.

We start this section by specifying, what well-defined elements in $T L_{n}(q)$ are. With more caution than in Section 3.1 we try to examine the relation between various path idempotents and also new idempotents, which are not necessarily path idempotents. This is done in Section 3.2.1. After that it is possible to give in Section 3.2.2 a description of the minimal central idempotents modulo the Jacobson radical of $T L_{n}(v)$, which are idempotents, that are central up to elements in the radical and minimal with that property. This is the main result of this section.

The statements in this section can be found in (GW93). Although our proofs are motivated by those in (GW93), in some cases they are not entirely the same. Infact, we only use diagramatic arguments. However, we stick to the naming of the statements given in (GW93).

We start by clarifying, what is meant by evaluable. Therefore let $\mathbb{C}[v]_{q}$ denote the subring of $\mathbb{C}(v)$ defined by

$$
\mathbb{C}[v]_{q}:=\left\{\frac{f}{g} ; f, g \in \mathbb{C}[v], g(q) \neq 0\right\}
$$

which is the subring of all rational functions, which can be evaluated at $v=q$.
Definition 3.2.1. $\left(T L_{n}\right)_{q}$ is defined to be $\mathbb{C}$-subspace of $T L_{n}$ consisting of the $\mathbb{C}[v]_{q^{-}}$ span of words in $U_{1}, \ldots, U_{n-1}$. The elements in $\left(T L_{n}\right)_{q}$ are called evaluable at $q$.

Since $T L_{n}$ is also a $\mathbb{C}$-algebra, and $\left(T L_{n}\right)_{q}$ is closed under multiplication, $\left(T L_{n}\right)_{q}$ is actually a unital $\mathbb{C}$-subalgebra of $T L_{n}$. The next proposition, which is actually (GW93, Proposition 0.1) and which we will not prove here, justifies Definition 3.2.1:
Proposition 3.2.2. The map defined from $\left(T L_{n}\right)_{q}$ to $T L_{n}(q)$ by sending $U_{i}$ to $U_{i}$ and $v$ to $q$ induces a surjective morphism of $\mathbb{C}$-algebras.

The morphism above is called the evaluation morphism and for an evaluable element $x \in T L_{n}, x(q)$ denotes the image of $x$ in $T L_{n}(q)$. The following lemma, needed for later results in Section 3.2.2, relates dimensions between $T L_{n}(q)$ and $T L_{n}$; it can also be found in (GW93, Proposition 0.1).
Lemma 3.2.3. If e and $f$ are evaluable idempotents in $T L_{n}$, then

$$
\operatorname{dim}_{\mathbb{C}}\left(e T L_{n} f\right)(q)=\operatorname{dim}_{\mathbb{C}(v)} e T L_{n} f
$$

Proof. Assume that there is a linear relation in $e T L_{n} f$ given by

$$
\gamma_{1} b_{1}+\cdots+\gamma_{k} b_{k}=0
$$

where $\gamma_{i} \in \mathbb{C}(v)$ and where the $b_{i}$ are words in $U_{1}, \ldots, U_{n-1} \in T L_{n}$. By multiplying with a sufficiently high power of $(v-q)$ a relation of the form

$$
\begin{equation*}
\gamma_{1}^{\prime} b_{1}+\cdots+\gamma_{k}^{\prime} b_{k}=0 \tag{3.20}
\end{equation*}
$$

is obtained such that $\gamma_{i}^{\prime} \in \mathbb{C}(v)_{q}$. This is actually a linear relation in $e\left(T L_{n}\right)_{q} f$. Without loss of generality, at least one coefficient $\gamma_{i}^{\prime}$ can be assumed not to be divisible by $(v-q)$, since otherwise dividing (3.20) by $(v-q)$ would still yield a linear relation of the above form in $e\left(T L_{n}\right)_{q} f$. Therefore, evaluating (3.20) at $v=q$ results in a non-trivial linear relation in $\left(e T L_{n} f\right)(q)$, which means that

$$
\begin{equation*}
\operatorname{dim}_{\mathbb{C}}\left(e T L_{n} f\right)(q) \leq \operatorname{dim}_{\mathbb{C}(v)} e T L_{n} f \tag{3.21}
\end{equation*}
$$

But (3.21) does not only hold for the idempotents $e$ and $f$, but also for the pairs of idempotents $(1-e, f),(e, 1-f)$ and $(1-e, 1-f)$. In particular, (3.21) implies

$$
\begin{align*}
\operatorname{dim}_{\mathbb{C}} T L_{n}(q)= & \operatorname{dim}_{\mathbb{C}}\left(e T L_{n} f\right)(q)+\operatorname{dim}_{\mathbb{C}}\left(e T L_{n}(1-f)\right)(q)+\operatorname{dim}_{\mathbb{C}}\left((1-e) T L_{n} f\right)(q) \\
& +\operatorname{dim}_{\mathbb{C}}\left((1-e) T L_{n}(1-f)\right)(q) \\
\leq & \operatorname{dim}_{\mathbb{C}(v)} e T L_{n} f+\operatorname{dim}_{\mathbb{C}(v)} e T L_{n}(1-f)+\operatorname{dim}_{\mathbb{C}(v)}(1-e) T L_{n} f \\
& +\operatorname{dim}_{\mathbb{C}(v)}(1-e) T L_{n}(1-f) \\
= & \operatorname{dim}_{\mathbb{C}(v)} T L_{n} \tag{3.22}
\end{align*}
$$

If (3.21) was strict, also (3.22) would be strict. But by (3.2.2) the inequality (3.22) actually has to be an equality, yielding that (3.21) needs to be an equality.

Now that we know, what evaluable elements are, the next step is to obtain an understanding of relations between idempotents. The following definition is quite standard:

Definition 3.2.4. Two idempotents $f$ and $e$ in a ring $R$ are called equivalent, if there exist elements $u, v \in R$, such that $e=u v$ and $f=v u$.

As the name suggests, this is an equivalence relation. This definition is justified by the following lemma, which is left as an exercise to the reader.

Lemma 3.2.5. Two idempotents $e$ and $f$ in a ring $R$ are equivalent, if and only if $R e \cong R f$ as left $R$-modules.

With this in mind, following (GW93), we refine the definition of equivalence:
Definition 3.2.6. Two evaluable idempotents $f, e \in T L_{n}$ are called equivalent, if there exist evaluable elements $u, v \in T L_{n}$, such that $e=u v$ and $f=v u$.

Often it is necessary to consider a set of pairwise equivalent and evaluable idempotens. Such a set actually corresponds to a so-called system of matrix units:

Remark 3.2.7. Let $u_{1}, \ldots, u_{k}$ be a set of evaluable pairwise equivalent idempotents and let the elements $u_{1 i}$ and $u_{i 1}$ implement the equivalence between $u_{1}$ and $u_{i}$, i.e. assume that $u_{1 i} u_{i 1}=u_{1}$ and $u_{i 1} u_{1 i}=u_{i}$. By setting $u_{i j}=u_{i} u_{i 1} u_{1 j} u_{j}$, the set $\left\{u_{i j}\right\}_{1 \leq i, j \leq k}$ actually satisfies the property

$$
u_{i j} u_{j i}=u_{i}=: u_{i i} \text { for all } 1 \leq i, j \leq k
$$

Moreover, it is easy to check, that one can actually assume

$$
\begin{equation*}
u_{i j} u_{j l}=u_{i l} \text { for all } 1 \leq i, j, l \leq k \tag{3.23}
\end{equation*}
$$

A set of evaluable elements $\left\{u_{i j}\right\}$ satisfying (3.23) is called a set of evaluable matrix units. Of course this is motivated by the matrix units $E_{i j} \in M_{n} \mathbb{C}$.

As now the termininology concerning evaluability and equivalence is specified, before proving a bunch of technical statements in Section 3.2.1, a last definition is needed. The next definition, following (GW93, Section 1), will be used to identify the future evaluable path idempotents:

Definition 3.2.8. 1. A partition $\lambda \in \operatorname{Par}_{2}(n)$ is called critical, if $w(\lambda):=\lambda_{1}-$ $\lambda_{2}+1$ is divisible by $l$ and the mth critical line in the branching graph is the line consisting of all partitions $\lambda$, with $w(\lambda)=m l$. Likewise a tableau $t$ of shape $\lambda$ is called critical, if $\lambda$ is critical. We refer to the largest proper critical subtableau of $t$ (if it exists) as the critical subtableau of $t$.
2. If a tableau has its critical subtableau $r$ on the mth critical line, then its conjugate $\bar{t}$ is defined to be the tableau corresponding to the path obtained from $t$ by fixing $r$ and reflecting $t \backslash r$ at the mth critical line. For any tableau the element $p_{[t]}$ is defined to be $p_{[t]}=p_{t}+p_{\bar{t}}$, where $p_{\bar{t}}:=0$, if $\bar{t}$ does not exist.
3. A critical tableau $t$ is called evaluable, if $p_{t}$ is evaluable and a non-critical tableau $t$ is called evaluable, if $p_{[t]}$ is evaluable.
4. A tableau t is called regular, if any two successive critical diagrams on $t$ are on different critical lines.

There is only one instance, where $t$ has a proper critical subtableau, but $\bar{t}$ does not exists, namely if the critical subtableau ends on the first critical line and $t$ ends on the second.

Example 3.2.9. The branching graph with vertical dashed lines as critical lines for $l=5$ is illustrated in Figure 7, together with a critical regular tableau drawn by dashed lines, an extension of that critical regular tableau and its conjugate.


Figure 7: A path and its conjugate.
Now the technical part of this thesis starts.

Section 3.2.1
Evaluable idempotents
This section mainly contains statements to prove Proposition 3.2.26 in the next section. Our goals are

1. to show that regular critical tableaux are evaluable (Proposition 3.2.17) and
2. to construct new evaluable idempotents out of old ones, see Lemma 3.2.12, Lemma 3.2.14 and Lemma 3.2.18,
while keeping track of equivalence relations between the involved idempotents. As already mentioned, the proofs are often motivated by those in (GW93, Section 1) for the corresponding statements. However, the proofs presented here do not use the representation theory of $T L_{n}$ but diagramatic arguements, which makes most them more technical.

We start by regarding a "baby" case:
Proposition 3.2.10. Let $r=\left((1)=\lambda^{(1)} \rightarrow \cdots \rightarrow \lambda^{(n)}=\lambda\right.$ ) be a tableau with no proper critical subtableau, i.e. $w\left(\lambda^{(i)}\right)<l$ for all $i$. Then $r$ is evaluable. Moreover, if $\tilde{r}$ is also such a tableau of same shape $\lambda$, then $p_{r}$ and $p_{\tilde{r}}$ are equivalent.

Proof. We use induction over $n$ and use Lemma 3.1.20 to show that $p_{r}$ is evaluable. For $n=1$, the path idempotent is just $1 \in T L_{1}$, so there is nothing to show.

1. Suppose that $\epsilon_{n-1}(r)=1$. By induction the idempotent $p_{r^{\prime}}$ is evaluable and non-zero. But Lemma 3.1.20 implies then

$$
p_{r}=\frac{[k-1]}{[k]} p_{r^{\prime}} U_{n-1} p_{r^{\prime}} \text { or } p_{r}=p_{r^{\prime}}-\frac{[k-1]}{[k]} p_{r^{\prime}} U_{n-1} p_{r^{\prime}}
$$

where $k=\lambda_{1}^{(n-1)}-\lambda_{2}^{(n-1)}+1=w\left(\lambda^{(n-1)}\right)$. Thus $p_{r}$ is evaluable. Since $p_{r^{\prime}}$ is non-zero by induction hypothesis and since $p_{r^{\prime}} U_{n-1} p_{r^{\prime}}$ and $p_{r^{\prime}}$ have different through-degrees in $T L_{n}$, also $p_{r}$ is non-zero.
2. If $\epsilon_{n-1}(r)=-1$, the second case of Lemma 3.1.20 applies, thus

$$
p_{r}=\frac{[k+1]}{[k]} p_{r^{\prime}} U_{n-1} p_{r^{\prime}} \text { or } p_{r}=p_{r^{\prime}}-\frac{[k+1]}{[k]} p_{r^{\prime}} U_{n-1} p_{r^{\prime}}
$$

where $k=\lambda_{1}^{(n-1)}-\lambda_{2}^{(n-1)}+1=\lambda_{1}^{(n-2)}-\lambda_{2}^{(n-2)}$, hence $k+1=w\left(\lambda^{(n-2)}\right)<l$. So again, $p_{r}$ stays evaluable, supposed $p_{r^{\prime}}$ is evaluable and similar as in the first case $p_{r}$ is also non-zero if $p_{r^{\prime}}$ is non-zero.

What is left is to show equivalence between two tableaux satisfying the assumptions. But if $t$ and $s$ are two evaluable tableaux of the same shape $\lambda$, it actually suffices to consider the case $s=s_{i}(t)$ for some admissible transposition $s_{i}$, since being equivalent is an equivalence relation, hence transitive. Moreover, only the case $i=n-1$, where $n=\lambda_{1}+\lambda_{2}$ is treated here, the general case follows from this one. The situation is illustrated in Figure 8. Let $r=t^{\prime \prime}$ and let $p_{r}, p_{s}$ and $p_{t}$ with


Figure 8: A path $t \in \operatorname{Std}(\lambda)$ and $s=s_{n-1}(t)$.
Definition 3.1.7 be given by


Now, if $u=\frac{[d+1]}{[d]} p_{s} U_{i} p_{t}$ and $w=\frac{[d+1]}{[d+2]} p_{t} U_{i} p_{s}$, then $u w=\frac{[d+1]^{2}}{[d+2][d]} p_{s} U_{i} p_{t} U_{i} p_{s}$ is of the form

where we used that $p_{d}$ is idempotent, (3.4) and Remark 3.1.11. One can also show that $w u=p_{t}$ by a similar argument. Furthermore both $u$ and $w$ are evaluable, since $p_{t}$ and $p_{s}$ are, and since $s$ and $t$ are to the left of the first critical line, which means that $[d],[d+1]$ and $[d+2]$ are non-zero and evaluable.

The previous lemma just states, that to the left of the first critical line everything is normal as in the generic case. Actually, the proof has the following consequence, which could be stated early in this theorem:

Corollary 3.2.11. If $t$ and $s$ are of same shape, then $p_{t} T L_{n} p_{s}$ is one dimensional.
Proof. In the proof of Proposition 3.2.10 a non-zero element $u \in p_{t} T L_{n} p_{s}$ was constructed, in particular, $p_{t} T L_{n} p_{s}$ is at least of dimension one. But since $T L_{n}=$ $\bigoplus_{t, s} p_{t} T L_{n} p_{s}$ and since the dimension of $T L_{n}$ is given by the number of pairs of paths of same shape (see Remark 2.2.6), this already implies that $p_{t} T L_{n} p_{s}$ must be of dimension one.

Now the first problems are to expect at the first critical line. The next lemma (cf. (GW93, Little Diamond Lemma)) states the existence of new evaluable idempotents which are not path idempotents.

Lemma 3.2.12 (Little Diamond Lemma). Let $r$ be an evaluable critical tableau of shape $\mu \in \operatorname{Par}_{2}(k)$ and set $\lambda=\left(\mu_{1}+1, \mu_{2}+1\right)$ and $d=\mu_{1}-\mu_{2}$. Let $s$ be the unique tableau ending in $\lambda$ and extending $r$, such that $t=s_{k}(s)$ is standard (see Figure 9). Then there exists a $3 \times 3$-system of evaluable matrix units in $T L_{k+2}$ with diagonal units equal to $[2] p_{t}, p_{r} U_{k+1}$ and $p_{r}\left([2]-U_{k+1}\right) p_{k+2, d}=p_{r}\left([2]-U_{k+1}\right) p_{[s]}$. In particular, $p_{r} p_{k+2, d}=p_{[s]}$ is evaluable.

We warn the reader, to proof consists of some tedious calculations.
Proof. As done often before, by abuse of notation we write $p_{r}$ instead $p_{r} \sqcup 1$. Moreover, let $\bar{s}=s_{k+1}(s)$. We will prove the lemma only for the first picture in


Figure 9: Paths $t, s=s_{k}(t)$ and $\bar{s}=s_{k+1}(s)$.
Figure 9, the second case works analoguously.
The first thing to do is showing that [2] $p_{t}$ and $p_{r} U_{k+1}$ are equivalent; therefore let $u$ and $w$ be defined by

$$
u=\left(1-p_{r}\right) U_{k} U_{k+1} p_{r} \text { and } w=p_{r} U_{k+1} U_{k}\left(1-p_{r}\right)
$$

They are both evaluable, since $p_{r}$ is evaluable. To show equivalence between [2] $p_{t}$ and $p_{r} U_{k+1}$ it is sufficient to prove the following equations:

$$
\begin{align*}
& w u=\frac{[d-1]}{[d]} p_{r} U_{k+1},  \tag{3.24}\\
& u w=[2] \frac{[d-1]}{[d]} p_{t} . \tag{3.25}
\end{align*}
$$

Proof of the equations (3.24) and (3.25). Starting with (3.24), Lemma 3.1.20 and the fact that $U_{k+1}$ commutes with $p_{r} \in T L_{k}$ imply

$$
\begin{equation*}
U_{k+1} p_{s^{\prime}} U_{k+1}=\frac{[d]}{[d+1]} U_{k+1} p_{r} U_{k} p_{r} U_{k+1}=\frac{[d]}{[d+1]} p_{r} U_{k+1} p_{r} \tag{3.26}
\end{equation*}
$$

This and Lemma 3.1.20 make

$$
\begin{align*}
p_{r} U_{k+1} U_{k} p_{r} U_{k} U_{k+1} p_{r} & =U_{k+1} p_{r} U_{k} p_{r} p_{r} U_{k} p_{r} U_{k+1}=\frac{[d+1]^{2}}{[d]^{2}} U_{k+1} p_{s^{\prime}} U_{k+1} \\
& =\frac{[d+1]}{[d]} p_{r} U_{k+1} p_{r}=\frac{[d+1]}{[d]} p_{r} U_{k+1} p_{r} \tag{3.27}
\end{align*}
$$

hold. Moreover, (2.5) has as consequence

$$
\begin{equation*}
p_{r} U_{k+1} U_{k} U_{k} U_{k+1} p_{r}=[2] p_{r} U_{k+1} U_{k} U_{k+1} p_{r}=[2] p_{r} U_{k+1} \tag{3.28}
\end{equation*}
$$

and therefore, it follows from (3.27) and (3.28), that

$$
\begin{aligned}
w u & =p_{r} U_{k+1} U_{k}\left(1-p_{r}\right) U_{k} U_{k+1} p_{r}=p_{r} U_{k+1} U_{k} U_{k} U_{k+1} p_{r}-p_{r} U_{k+1} U_{k} p_{r} U_{k} U_{k+1} p_{r} \\
& =[2] p_{r} U_{k+1}-\frac{[d+1]}{[d]} p_{r} U_{k+1}=\frac{[d-1]}{[d]} p_{r} U_{k+1},
\end{aligned}
$$

which is just (3.24).
Now let $\omega=t^{\prime \prime}$. Since $p_{r} p_{r^{\prime}}=p_{r}$ and $p_{r^{\prime}}-p_{r}=p_{\omega}$ by (3.6) and moreover, since $p_{r^{\prime}} \in T L_{k-1}$ commutes with $U_{k}$ and $U_{k+1}$, we obtain

$$
\begin{align*}
u w & =\left(1-p_{r}\right) U_{k} U_{k+1} p_{r} U_{k+1} U_{k}\left(1-p_{r}\right)=[2]\left(1-p_{r}\right) U_{k} p_{r^{\prime}} p_{r} p_{r^{\prime}} U_{k+1} U_{k}\left(1-p_{r}\right) \\
& =[2]\left(p_{r^{\prime}}-p_{r}\right) U_{k} p_{r} U_{k+1} U_{k}\left(p_{r^{\prime}}-p_{r}\right)=[2] p_{\omega} U_{k} p_{r} U_{k+1} U_{k} p_{\omega} \\
& =[2] p_{\omega} U_{k}\left(p_{r^{\prime}}-p_{\omega}\right) U_{k+1} U_{k} p_{\omega} \\
& =[2] p_{\omega} U_{k} U_{k+1} U_{k} p_{\omega}-[2] p_{\omega} U_{k} p_{\omega} U_{k+1} p_{\omega} U_{k} p_{\omega} \\
& =[2] p_{\omega} U_{k} p_{\omega}-[2] p_{\omega} U_{k} p_{\omega} U_{k+1} p_{\omega} U_{k} p_{\omega} . \tag{3.29}
\end{align*}
$$

Lemma 3.1.20 implies $p_{\omega} u_{k} p_{\omega}=\frac{[d-1]}{[d]} p_{t^{\prime}}$ and therefore from (3.29) it follows

$$
\begin{aligned}
u w & =[2] \frac{[d-1]}{[d]} p_{t^{\prime}}-[2] \frac{[d-1]^{2}}{[d]^{2}} p_{t^{\prime}} U_{k+1} p_{t^{\prime}}=[2] \frac{[d-1]}{[d]}\left(p_{t^{\prime}}-\frac{[d-1]}{[d]} p_{t^{\prime}} U_{k+1} p_{t^{\prime}}\right) \\
& =[2] \frac{[d-1]}{[d]} p_{t},
\end{aligned}
$$

where Lemma 3.1.20 was used for the last equality. But this is just (3.25).
Now [2] $p_{t}$ and $p_{r}\left([2]-U_{k+1}\right) p_{n, k}$ are known to be equivalent. With

$$
\begin{equation*}
\hat{w}=\left([2]-U_{k+1}\right) p_{r} U_{k}\left(1-p_{r}\right) p_{t} \text { and } \hat{u}=p_{t}\left(1-p_{r}\right) U_{k}\left([2]-U_{k+1}\right) p_{r}, \tag{3.30}
\end{equation*}
$$

the elements [2] $p_{t}$ and $p_{r}\left([2]-U_{k+1}\right) p_{n, k}$ will be equivalent by showing the following equations:

$$
\begin{align*}
& \hat{u} \hat{w}=[2] \frac{[d-1][d+2]}{[d]^{2}} p_{t},  \tag{3.31}\\
& \hat{w} \hat{u}=\frac{[d-1][d+2]}{[d]^{2}} p_{r}\left([2]-U_{k+1}\right) p_{n, k} . \tag{3.32}
\end{align*}
$$

Proof of the equations (3.31) and (3.32). Lemma 3.1.20 implies that

$$
\begin{align*}
U_{k} p_{r} U_{k} & =U_{k}\left(p_{r^{\prime}}-\frac{[d-1]}{[d]} p_{r^{\prime}} U_{k-1} p_{r^{\prime}}\right) U_{k}=[2] U_{k} p_{r}^{\prime}-\frac{[d-1]}{[d]} p_{r^{\prime}} U_{k} \\
& =\frac{[d+1]}{[d]} p_{r^{\prime}} U_{k} \tag{3.33}
\end{align*}
$$

and Lemma 3.1.20 that

$$
\begin{equation*}
p_{\omega} U_{k} p_{\omega}=\frac{[d-1]}{[d]} p_{t^{\prime}} . \tag{3.34}
\end{equation*}
$$

So together (3.33) and (3.34) yield

$$
\begin{align*}
p_{t}\left(1-p_{r}\right) U_{k} p_{r} U_{k}\left(1-p_{r}\right) p_{t} & =p_{t} p_{\omega} U_{k} p_{r} U_{k} p_{\omega} p_{t}=\frac{[d+1]}{[d]} p_{t} p_{\omega} U_{k} p_{r^{\prime}} p_{\omega} p_{t} \\
& =\frac{[d+1][d-1]}{[d]^{2}} p_{t} p_{t^{\prime}} p_{t}=\frac{[d+1][d-1]}{[d]^{2}} p_{t}, \tag{3.35}
\end{align*}
$$

where we used the second part of Remark 3.1.8 for $t, t^{\prime}, \omega=t^{\prime \prime}$ and $r^{\prime}=\omega^{\prime}$. Moreover, $p_{r}$ commuting with [2] $-U_{k+1}$ and ([2] $\left.-U_{k+1}\right)^{2}=[2]\left([2]-U_{k+1}\right)$ imply

$$
\begin{aligned}
\hat{u} \hat{w} & =p_{t}\left(1-p_{r}\right) U_{k}\left([2]-U_{k+1}\right) p_{r}\left([2]-U_{k+1}\right) p_{r} U_{k}\left(1-p_{r}\right) p_{t} \\
& =[2]\left(p_{t}\left(1-p_{r}\right) U_{k}\left([2]-U_{k+1}\right) p_{r} U_{k}\left(1-p_{r}\right) p_{t}\right) \\
& =[2]\left([2] p_{t}\left(1-p_{r}\right) U_{k} p_{r} U_{k}\left(1-p_{r}\right) p_{t}-p_{t}\left(1-p_{r}\right) U_{k} U_{k+1} p_{r} U_{k}\left(1-p_{r}\right) p_{t}\right) .
\end{aligned}
$$

Substituting (3.35) and (3.25) shows now

$$
\begin{aligned}
\hat{u} \hat{w} & =[2]\left([2] \frac{[d-1][d+1]}{[d]^{2}} p_{t}-\frac{1}{[2]} u w\right)=[2]\left([2] \frac{[d-1][d+1]}{[d]^{2}} p_{t}-\frac{[d-1]}{[d]} p_{t}\right) \\
& =[2] \frac{[d-1][d+2]}{[d]^{2}} p_{t} .
\end{aligned}
$$

which is (3.31). The last equation to show is (3.32). Applying (3.11) and using $p_{t}=p_{t}\left(p_{s}+p_{t}\right)$ results in the equality

$$
\begin{align*}
p_{t} U_{k}\left([2]-U_{k+1}\right) p_{r} & =p_{t}\left(p_{s}+p_{t}\right) U_{k}\left([2]-U_{k+1}\right) p_{r}=p_{t} U_{k}\left(p_{s}+p_{t}\right)\left([2]-U_{k+1}\right) p_{r} \\
& =p_{t} U_{k} p_{s}\left([2]-U_{k+1}\right) p_{r} . \tag{3.36}
\end{align*}
$$

since $p_{t} U_{k+1}=0$ by 3.7 and since $p_{t} p_{r}=0$, because $t$ is not an extension of $r$. The equations $p_{s}=p_{s}\left(p_{s}+p_{\bar{s}}\right),(3.11)$ and (3.36) imply together

$$
\begin{align*}
p_{t} U_{k}\left([2]-U_{k+1}\right) p_{r} & =p_{t} U_{k} p_{s}\left([2]-U_{k+1}\right) p_{r}=p_{t} U_{k}\left(p_{s}+p_{\bar{s}}\right)\left([2]-U_{k+1}\right) p_{r} \\
& =p_{t} U_{k}\left([2]-U_{k+1}\right)\left(p_{s}+p_{\bar{s}}\right) p_{r} \\
& =p_{t} U_{k}\left([2]-U_{k+1}\right) p_{r} p_{k+2, d} \tag{3.37}
\end{align*}
$$

since $p_{k+2, d}=\sum_{\sigma \in \operatorname{Std}(\lambda)} p_{\sigma}$ by Definition 3.1.22. Observe also that

$$
\begin{equation*}
p_{r^{\prime}}\left(1-p_{r}\right) p_{k+2, d}=p_{\omega} p_{k+2, d}=p_{t^{\prime}} p_{k+2, d}=p_{t} p_{k+2, d} \tag{3.38}
\end{equation*}
$$

which implies with (3.37) and the fact that $p_{k+2, d}$ is central idempotent, that

$$
\begin{align*}
p_{t} U_{k}\left([2]-U_{k+1}\right) p_{r} & =p_{t} U_{k}\left([2]-U_{k+1}\right) p_{r} p_{k+2, d} \\
=p_{t} p_{k+2, d} U_{k}\left([2]-U_{k+1}\right) p_{r} p_{k+2, d} & \\
& =p_{r^{\prime}}\left(1-p_{r}\right) p_{k+2, d} U_{k}\left([2]-U_{k+1}\right) p_{r} p_{k+2, d} \\
& =p_{\omega} U_{k}\left([2]-U_{k+1}\right) p_{r} p_{k+2, d} . \tag{3.39}
\end{align*}
$$

Similar to (3.26), Lemma 3.1.20 implies

$$
\begin{equation*}
U_{k} p_{\omega} U_{k}=U_{k}\left(\frac{[d-1]}{[d]} p_{r^{\prime}} U_{k-1} p_{r^{\prime}}\right) U_{k}=\frac{[d-1]}{[d]} p_{r^{\prime}} U_{k} . \tag{3.40}
\end{equation*}
$$

Then (3.40) and (3.39) together yield

$$
\begin{aligned}
U_{k} p_{t} U_{k}\left([2]-U_{k+1}\right) p_{r} & =U_{k} p_{\omega} U_{k}\left([2]-U_{k+1}\right) p_{r} p_{k+2, d} \\
& =\frac{[d-1]}{[d]} p_{r^{\prime}} U_{k}\left([2]-U_{k+1}\right) p_{r} p_{k+2, d} \\
& =\frac{[d-1]}{[d]} p_{r^{\prime}} U_{k} p_{r}\left([2]-U_{k+1}\right) p_{r} p_{k+2, d},
\end{aligned}
$$

which also implies with Lemma 3.1.20 that

$$
\begin{align*}
p_{r} U_{k} p_{t} U_{k}\left([2]-U_{k+1}\right) p_{r} & =\frac{[d-1]}{[d]} p_{r} U_{k} p_{r}\left([2]-U_{k+1}\right) p_{r} p_{k+2, d} \\
& =\frac{[d-1][d+1]}{[d]^{2}} p_{s^{\prime}}\left([2]-U_{k+1}\right) p_{r} p_{k+2, d} \\
& =\frac{[d-1][d+1]}{[d]^{2}} p_{s^{\prime}}\left([2]-U_{k+1}\right)\left(p_{s^{\prime}}+p_{\bar{s}^{\prime}}\right) p_{k+2, d} . \tag{3.41}
\end{align*}
$$

Lemma 3.1.20 also implies

$$
\begin{equation*}
p_{s^{\prime}}\left([2]-U_{k+1}\right) p_{s^{\prime}} p_{k+2, d}=\left([2]-\frac{[d]}{[d+1]}\right) p_{s} p_{k+2, d}=\frac{[d+2]}{[d+1]} p_{s} p_{k+2, d} \tag{3.42}
\end{equation*}
$$

Moreover, with $p_{\bar{s}} U_{k+1} p_{\bar{s}}=-p_{s} U_{k+1} p_{\bar{s}}+U_{k+1} p_{s}$ from (3.11), Lemma 3.1.20 gives

$$
\begin{align*}
& \left([2]-U_{k+1}\right) p_{s^{\prime}}\left([2]-U_{k+1}\right) p_{\bar{s}^{\prime}} p_{k+2, d} \\
& =\left([2]-U_{k+1}\right) p_{s}\left([2]-U_{k+1}\right) p_{\bar{s}} p_{k+2, d}=-\left([2]-U_{k+1}\right) p_{s} U_{k+1} p_{\bar{s}} p_{k+2, d} \\
& =\left([2]-U_{k+1}\right) p_{\bar{s}} U_{k+1} p_{\bar{s}} p_{k+2, d}+\left([2]-U_{k+1}\right) U_{k+1} p_{s} \\
& =\left([2]-U_{k+1}\right) p_{\bar{s}^{\prime}} U_{k+1} p_{\bar{s}^{\prime}} p_{k+2, d}=\left([2]-U_{k+1}\right) \frac{[d+2]}{[d+1]} p_{\bar{s}} p_{k+2, d} . \tag{3.43}
\end{align*}
$$

Now we can deduce from (3.30), (3.41), (3.42) and (3.43) that

$$
\begin{aligned}
\hat{w} \hat{u} & =\left([2]-U_{k+1}\right) p_{r} U_{k} p_{t} U_{k}\left([2]-U_{k+1}\right) p_{r} p_{k+2, d} \\
& =\frac{[d-1][d+1]}{[d]^{2}}\left([2]-U_{k+1}\right) p_{s^{\prime}}\left([2]-U_{k+1}\right)\left(p_{s^{\prime}}+p_{\bar{s}^{\prime}}\right) p_{k+2, d} \\
& =\frac{[d-1][d+1]}{[d]^{2}}\left([2]-U_{k+1}\right)\left(\frac{[d+2]}{[d+1]} p_{s} p_{k+2, d}+\frac{[d+2]}{[d+1]} p_{\bar{s}} p_{k+2, d}\right) \\
& =\frac{[d-1][d+2]}{[d]^{2}}\left([2]-U_{k+1}\right)\left(p_{s}+p_{\bar{s}}\right) p_{k+2, d},
\end{aligned}
$$

which implies with $\left(p_{s}+p_{\bar{s}}\right) p_{k+2, d}=p_{[s]} p_{k+2, d}=p_{r} p_{k+2, d}$ that

$$
\hat{w} \hat{u}=\frac{[d-1][d+2]}{[d]^{2}}\left([2]-U_{k+1}\right) p_{r} p_{k+2, d}=\frac{[d-1][d+2]}{[d]^{2}} p_{r}\left([2]-U_{k+1}\right) p_{k+2, d},
$$

so finally (3.32) is shown.
Since [2] $p_{t}$ and $p_{r}\left([2]-U_{k+1}\right) p_{[s]}$ are now equivalent ((3.24), (3.25),(3.31) and (3.32)) and evaluable ( $u, w, \hat{u}$ and $\hat{w}$ were evaluable), also the sum

$$
p_{r}\left([2]-U_{k+1}\right) p_{[s]}+p_{r} U_{k+1}=p_{r}[2] p_{k+2, d}=p_{r}[2] p_{[s]}
$$

is evaluable and hence $p_{r} p_{k+2, d}$ is.
In Lemma 3.2.12, we saw evaluability of $p_{[s]}$, where $s=r^{+-}$for a evaluable critical tableau $r$. Now the next lemma (cf. (GW93, Interpolation Lemma)) generalizes this statement; it states that whenever the critical subtableau is evaluable, also $p_{[s]}$ is so. As the previous proof, the following proof is rather calculation heavy.

Lemma 3.2.13 (Interpolation lemma).
Let t be a tableau with evaluable critical subtableau $r$.

1. $p_{[t]}$ is evaluable. In particular, if $r$ ends on the first critical line and $t$ on the second critical line, then $p_{t}=p_{[t]}$ is evaluable.
2. Let $s$ be another tableau with same critical subtableau and same shape as $t$. Then $p_{[t]}$ and $p_{[s]}$ are equivalent.

Proof. Let $\emptyset \rightarrow(1)=\lambda^{(1)} \rightarrow \cdots \rightarrow \lambda^{(n)}$ be the path associated to $t$ and let $k \leq n$, such that the critical subtableau $r$ of $t$ is of shape $\mu=\lambda^{(k)}$. Assume that $r$ ends on the $m$ th critical line. Moreover, we can also assume that $\lambda^{(n)}$ is to the right of the $m$ th critical line (if not take $\bar{t}$ instead of $t$ ), i.e. $w\left(\lambda^{(n)}\right) \geq m l$. Furthermore, let $\bar{\lambda}=\operatorname{Shape}(\bar{t})$. We use induction over $n-k$ to prove the statement. Clearly the case $n-k=1$ holds, since then $p_{[t]}=p_{t}+p_{\bar{t}}=p_{r} \sqcup 1$ is evaluable, and moreover $n-k=2$ implies that

$$
p_{[t]}=(p r \sqcup 1) p_{k, d} \text { or } p_{[t]}=(p r \sqcup 1)\left(1-p_{k, d}\right),
$$

where $d=\lambda_{1}^{(n)}-\lambda_{2}^{(n)}=\mu_{1}+1-\mu_{2}-1=\mu_{1}-\mu_{2}$, thus for $n-k=2$ the statement follows from Lemma 3.2.12. Therefore $n-k$ is assumed to be at least 3 and $p_{\left[t^{\prime}\right]}$ to be evaluable by induction hypothesis. There are the following cases:

1. Assume that $\lambda^{(n)}=\left(\lambda_{1}^{(n-2)}+1, \lambda_{2}^{(n-2)}+1\right)$ and assume that $\epsilon_{n}(t)=-1$ : The situation is illustrated in Figure 10. Since $n-k \geq 3, \lambda^{(n-2)}$ is not critical, so $\bar{t} \neq s_{n-1}(t)$. Therefore, it follows from Corollary 3.1.19 that $p_{t^{\prime}} U_{n-1} p_{\bar{t}^{\prime}}=0$


Figure 10: A path $t$ and its conjugate $\bar{t}$ with $\epsilon_{n}(t)=-1$.
and moreover, defining the element $E$ by $E=p_{\left[t^{\prime}\right]} U_{n-1} p_{\left[t^{\prime}\right]}$, together with Lemma 3.1.20 yields

$$
E=p_{t^{\prime}} U_{n-1} p_{t^{\prime}}+p_{\bar{t}^{\prime}} U_{n-1} p_{\bar{t}^{\prime}}=\frac{[w(\lambda)+1]}{[w(\lambda)]} p_{t}+\frac{[w(\bar{\lambda})-1]}{[w(\bar{\lambda})]} p_{\bar{t}}
$$

Therefore (3.1.17) implies

$$
\begin{equation*}
p_{[t]}=\frac{\left(\frac{[w(\lambda)+1]}{[w(\lambda)]}+\frac{[w(\bar{\lambda})-1]}{[w(\bar{\lambda})]}\right) E-E^{2}}{\frac{[w(\lambda)+1]}{[w(\lambda)]} \frac{[w(\bar{\lambda})-1]}{[w(\bar{\lambda})]}} . \tag{3.44}
\end{equation*}
$$

$t^{\prime}$ being to the right of the critical line and $t$ beeing not on a critical line, together give $m l+l-1 \geq w\left(\lambda^{(n)}\right)=w\left(\lambda^{(n-2)}\right) \geq m l+2$ and $m l-l+1 \leq$ $w(\bar{\lambda}) \leq m l-2$. Therefore, (3.44) implies that $p_{[t]}$ is evaluable. The case $\epsilon_{n}(t)=1$ is similar.
2. If $\lambda^{(n)}=\left(\lambda_{1}^{(n-2)}+2, \lambda_{2}^{(n-2)}\right)$ or if $\lambda^{(n)}=\left(\lambda_{1}^{(n-2)}, \lambda_{2}^{(n-2)}+2\right)$, with $s$ defined as the unique extension of $t^{\prime}$ different from $t$, then the statement for $s$ follows by the previous case. But then $p_{[t]}=p_{\left[t^{\prime}\right]} \sqcup 1-p_{[s]}$ is also evaluable.

Now we turn our focus to showing equivalence. The path $s$ can be obtained form $t$ by applying a sequence of simple transpositions, as usual it suffices to assume $s=s_{i}(t)$ for some $i$. We further assume that $i=n-1$, the general case follows from this one. Setting

$$
u=p_{[s]} U_{i} p_{[t]} \text { and } u^{\prime}=p_{[t]} U_{i} p_{[s]},
$$

and using $p_{s} U_{i} p_{\bar{t}}=0=p_{\bar{s}} U_{i} p_{t}$ from Corollary 3.1.19, implies

$$
\begin{equation*}
u u^{\prime}=p_{s} U_{i} p_{t} U_{i} p_{s}+p_{\bar{s}} U_{i} p_{\bar{t}} U_{i} p_{\bar{s}} . \tag{3.45}
\end{equation*}
$$

By using Lemma 3.1.20 similar as for (3.26) and applying Lemma 3.1.20 again, we can deduce that

$$
\begin{align*}
& p_{s} U_{i} p_{t} U_{i} p_{s}=\frac{[w(\lambda)-1]}{[w(\lambda)]} \frac{[w(\lambda)+1]}{[w(\lambda)]} p_{s},  \tag{3.46}\\
& p_{\bar{s}} U_{i} p_{\bar{t}} U_{i} p_{\bar{s}}=\frac{[w(\bar{\lambda})-1]}{[w(\bar{\lambda})]} \frac{[w(\bar{\lambda})+1]}{[w(\bar{\lambda})]} p_{\bar{s}}, \tag{3.47}
\end{align*}
$$

where

$$
c=\frac{[w(\lambda)-1]}{[w(\lambda)]} \frac{[w(\lambda)+1]}{[w(\lambda)]} \text { and } \bar{c}=\frac{[w(\bar{\lambda})-1]}{[w(\bar{\lambda})]} \frac{[w(\bar{\lambda})+1]}{[w(\bar{\lambda})]} .
$$

Substituting (3.46) and (3.47) into (3.45) implies

$$
\begin{equation*}
u u^{\prime}=c p_{s}+\bar{c} p_{\bar{s}} . \tag{3.48}
\end{equation*}
$$

Similarly one can show that

$$
\begin{equation*}
u^{\prime} u=c p_{t}+\bar{c} p_{\bar{t}} . \tag{3.49}
\end{equation*}
$$

The equations (3.48) and (3.49) have as consequence the equations

$$
\begin{aligned}
\left(u u^{\prime}-\bar{c}\right) p_{\bar{s}}=0 & =\left(u u^{\prime}-c\right) p_{s}, \\
\left(u^{\prime} u-\bar{c}\right) p_{\bar{t}}=0= & =\left(u^{\prime} u-c\right) p_{t},
\end{aligned}
$$

and therefore, we obtain

$$
\begin{equation*}
\left(u^{\prime} u-c\right)\left(u^{\prime} u-\bar{c}\right) p_{[t]}=0=\left(u u^{\prime}-c\right)\left(u u^{\prime}-\bar{c}\right) p_{[s]} . \tag{3.50}
\end{equation*}
$$

Then (3.50) together with $u^{\prime} u p_{[s]}=0=u u^{\prime} p_{[t]}$, because of (3.48) and (3.49), give the two equations

$$
\begin{align*}
& \frac{\left(u^{\prime} u-c\right)\left(u^{\prime} u-\bar{c}\right)}{c \bar{c}}\left(p_{[s]}+p_{[t]}\right)=p_{[t]}  \tag{3.51}\\
& \frac{\left(u u^{\prime}-c\right)\left(u u^{\prime}-\bar{c}\right)}{c \bar{c}}\left(p_{[s]}+p_{[t]}\right)=p_{[s]} . \tag{3.52}
\end{align*}
$$

Setting $w=-\frac{u u^{\prime} u-(c+\bar{c}) u}{\bar{c}}$ with (3.51) and (3.52) imply that

$$
p_{[s]}=u^{\prime} w \text { and } p_{[t]}=w u^{\prime},
$$

which means that $p_{[s]}$ and $p_{[t]}$ are equivalent.
As before, also the next statement has a quite technical proof. It will be used to connect the evaluable idempotents by equivalence in the next section.

Lemma 3.2.14. Let $r$ be an evaluable critical tableau of shape $\mu \in \operatorname{Par}_{2}(k)$ and set $\lambda=\left(\mu_{1}+1, \mu_{2}+1\right)$. Consider the six extensions of $r$ of length $k+3$ which end in $\left(\mu_{1}+1, \mu_{2}+2\right)$ or in $\left(\mu_{1}+2, \mu_{1}+1\right)$ and let t denote the left most of those (see also Figure 11). Then there is a $3 \times 3$-system of evaluable matrix units in $T L_{k+3}$ with diagonal matrix units $[2] p_{[t]}, p_{r} U_{k+1}$ and $p_{r}\left([2]-U_{k+1}\right) p_{k+2, d}$ where $d=\mu_{1}-\mu_{2}=\lambda_{1}-\lambda_{2}$.

Proof. We are in the situation of Figure 11, where the path $r$ ends in $\mu, t$ is the left most path ending in $\left(\mu_{1}+1, \mu_{2}+2\right)$ and $\bar{t}$ is the right most path ending in $\left(\mu_{1}+2, \mu_{2}+1\right)$. By Lemma 3.2.13 $p_{[t]}$ is evaluable and by Lemma 3.2.12 $p_{r} U_{k+1}$


Figure 11: The path $r$ and its six extensions $t, s, w \bar{t}, \bar{s}$ and $\bar{w}$.
is evaluable too. Again by Lemma 3.2.12, it suffices to show equivalence between [2] $p_{[t]}$ and $p_{r} U_{k+1}$. Let $u$ and $u^{\prime}$ be defined by

$$
\begin{equation*}
u=p_{r} U_{k+1} U_{k+2} p_{[t]} \text { and } u^{\prime}=p_{[t]} U_{k+2} U_{k+1} p_{r} \tag{3.53}
\end{equation*}
$$

We first show the following two equations:

$$
\begin{align*}
& -\frac{u^{\prime} u u^{\prime} u-[2](c+\bar{c}) u^{\prime} u}{[2] c \bar{c}}=[2] p_{[t]},  \tag{3.54}\\
& -\frac{u u^{\prime} u u^{\prime}-[2](c+\bar{c}) u u^{\prime}}{[2] c \bar{c}}=p_{r} U_{k+1}, \tag{3.55}
\end{align*}
$$

where $c=\frac{[d-1]}{[d]}$ and $\bar{c}=\frac{[d+3]}{[d+2]}$ are non-zero and evaluable, since $d+1=w(\mu)=m l$. Proof of the equations (3.54) and (3.55). Since $p_{r} \in T L_{k}$ commutes with $U_{k+1}$ and $U_{k+2}$ and since $p_{r} p_{[t]}=p_{[t]}$, the following holds:

$$
\begin{align*}
u^{\prime} u & =p_{[t]} U_{k+2} U_{k+1} p_{r} p_{r} U_{k+1} U_{k+2} p_{[t]}=p_{[t]} p_{r} U_{k+2} U_{k+1} U_{k+1} U_{k+2} p_{[t]} \\
& =[2] p_{[t]} U_{k+2} U_{k+1} U_{k+2} p_{[t]}=[2] p_{[t]} U_{k+2} p_{[t]} . \tag{3.56}
\end{align*}
$$

With Corollary 3.1.19, (3.56) actually refines to

$$
\begin{equation*}
u^{\prime} u=[2] p_{t} U_{k+2} p_{t}+[2] p_{\bar{t}} U_{k+2} p_{\bar{t}} . \tag{3.57}
\end{equation*}
$$

and applying Lemma 3.1.20 to $p_{t}$ and $p_{\bar{r}}$ in (3.57) yields

$$
\begin{equation*}
u^{\prime} u=[2]\left(c p_{t}+\bar{c} p_{\bar{t}}\right) . \tag{3.58}
\end{equation*}
$$

Using (3.58) implies

$$
\begin{aligned}
u^{\prime} u u^{\prime} u-[2](c+\bar{c}) u^{\prime} u & =[2]^{2}\left(c^{2} p_{t}+\bar{c}^{2} p_{\bar{t}}\right)-[2](c+\bar{c})[2]\left(c p_{t}+\bar{c} p_{\bar{t}}\right) \\
& =-[2]^{2} c \bar{c}\left(p_{t}+p_{\bar{t}}\right)=-[2]^{2} c \bar{c} p_{[t]}
\end{aligned}
$$

and moreover

$$
-\frac{u^{\prime} u u^{\prime} u-[2](c+\bar{c}) u^{\prime} u}{[2] c \bar{c}}=[2] p_{[t]},
$$

which is just (3.54). To show (3.55) we have to work a little more.
Since $U_{k+2}$ commutes with $p_{t^{\prime \prime}} \in T L_{k+1}$, Lemma 3.1.20 implies

$$
\begin{align*}
U_{k+2} p_{t^{\prime}} U_{k+2} & =U_{k+2}\left(p_{t^{\prime \prime}}-p_{t^{\prime \prime}} U_{k+1} p_{t^{\prime \prime}} \frac{[d+1]}{[d]}\right) U_{k+2} \\
& =U_{k+2} p_{t^{\prime \prime}} U_{k+2}-p_{t^{\prime \prime}} U_{k+2} U_{k+1} U_{k+2} p_{t^{\prime \prime}} \frac{[d+1]}{[d]} \\
& =[2] U_{k+2} p_{t^{\prime \prime}}-p_{t^{\prime \prime}} U_{k+2} p_{t^{\prime \prime}} \frac{[d+1]}{[d]}=U_{k+2} p_{t^{\prime \prime}}\left([2]-\frac{[d+1]}{[d]}\right) \\
& =U_{k+2} p_{t^{\prime \prime}} \frac{[d-1]}{[d]} . \tag{3.59}
\end{align*}
$$

By applying Lemma 3.1.20 and substituting (3.59) twice, one sees

$$
\begin{align*}
U_{k+2} p_{t} U_{k+2} & =\frac{[d]}{[d-1]} U_{k+2} p_{t^{\prime}} U_{k+2} p_{t^{\prime}} U_{k+2}=U_{k+2} p_{t^{\prime}} U_{k+2} p_{t^{\prime \prime}} \\
& =U_{k+2} p_{t^{\prime \prime}} \frac{[d-1]}{[d]} p_{t^{\prime \prime}}=\frac{[d-1]}{[d]} U_{k+2} p_{t^{\prime \prime}}=c U_{k+2} p_{t^{\prime \prime}} \tag{3.60}
\end{align*}
$$

Similarly the equation

$$
\begin{equation*}
U_{k+2} p_{\bar{t}} U_{k+2}=\bar{c} p_{\bar{t}^{\prime \prime}} U_{k+2} \tag{3.61}
\end{equation*}
$$

holds. Then (3.60) and (3.61) imply together

$$
\begin{align*}
u u^{\prime} & =p_{r} U_{k+1} U_{k+2}\left(p_{t}+p_{\bar{t}}\right) U_{k+2} U_{k+1} p_{r} \\
& =p_{r} U_{k+1} U_{k+2} p_{t} U_{k+2} U_{k+1} p_{r}+p_{r} U_{k+1} U_{k+2} p_{\bar{t}} U_{k+2} U_{k+1} p_{r} \\
& =c p_{r} U_{k+1} U_{k+2} p_{t^{\prime \prime}} U_{k+1} p_{r}+\bar{c} p_{r} U_{k+1} p_{\bar{t}^{\prime \prime}} U_{k+2} \bar{U}_{k+1} p_{r} \\
& =c U_{k+1} U_{k+2} p_{t^{\prime \prime}} U_{k+1}+\bar{c} U_{k+1} p_{t^{\prime \prime}} U_{k+2} \bar{U}_{k+1}, \tag{3.62}
\end{align*}
$$

where we also used, that $p_{t^{\prime \prime}} \in T L_{k+1}$ commutes with $U_{k+2}$ and that $p_{r} \in T L_{k}$ commutes with $U_{k+1}$. If $s$ and $w$ denote the other two extensions of $r$ ending in Shape $(t)$ (see Figure 11), then $s$ is an extension of $t^{\prime \prime}$ and $w$ of $\vec{t}^{\prime \prime}$. In particular,

$$
\begin{align*}
& p_{\bar{t}^{\prime \prime}} U_{k+2} U_{k+1}\left(p_{w}+p_{s}\right)=p_{\bar{t}^{\prime \prime}} U_{k+2}\left(p_{w}+p_{s}\right) U_{k+1}=p_{\bar{t}^{\prime \prime}} U_{k+2} p_{s} U_{k+1} \\
& =p_{\bar{t}^{\prime \prime}} U_{k+2}\left(p_{s}+p_{t}\right) p_{s} U_{k+1}=p_{\bar{t}^{\prime \prime}}\left(p_{s}+p_{t}\right) U_{k+2} p_{s} U_{k+1}=0 \tag{3.63}
\end{align*}
$$

holds by Lemma 3.1.18 and (3.7). (3.62) followed by (3.63) imply

$$
\begin{align*}
u u^{\prime} U_{k+1}\left(p_{w}+p_{s}\right) & =\left(c U_{k+1} U_{k+2} p_{t^{\prime \prime}} U_{k+1}+\bar{c} U_{k+1} p_{\bar{t}^{\prime \prime}} U_{k+2} U_{k+1}\right) U_{k+1}\left(p_{w}+p_{s}\right) \\
& =c U_{k+1} U_{k+2} p_{t^{\prime \prime}} U_{k+1} U_{k+1}\left(p_{w}+p_{s}\right) \\
& =[2] c U_{k+1} U_{k+2} p_{t^{\prime \prime}}\left(p_{s}+p_{w}\right) U_{k+1} \\
& =[2] c U_{k+1} U_{k+2}\left(p_{s}+p_{w}\right) U_{k+1} \\
& =[2] c U_{k+1} U_{k+2} U_{k+1}\left(p_{s}+p_{w}\right)=[2] c U_{k+1}\left(p_{s}+p_{w}\right), \tag{3.64}
\end{align*}
$$

where we also used Lemma 3.1.18. Similarly to (3.64), one can show that

$$
\begin{equation*}
u u^{\prime} U_{k+1}\left(p_{\bar{w}}+p_{\bar{s}}\right)=[2] \bar{c} U_{k+1}\left(p_{\bar{s}}+p_{\bar{w}}\right) \tag{3.65}
\end{equation*}
$$

holds. Now (3.53), (3.64), (3.65) and $U_{k+1} p_{[t]}=0$, because of (3.7), imply together

$$
\begin{aligned}
{[2] u u^{\prime} } & =u u^{\prime} U_{k+1} p_{r}=u u^{\prime} U_{k+1}\left(p_{w}+p_{s}\right)+u u^{\prime} U_{k+1}\left(p_{\bar{w}}+p_{\bar{s}}\right) \\
& =[2] c U_{k+1}\left(p_{s}+p_{w}\right)+[2] \bar{c} U_{k+1}\left(p_{\bar{s}}+p_{\bar{w}}\right)
\end{aligned}
$$

which implies that

$$
\begin{equation*}
u u^{\prime}=c U_{k+1}\left(p_{s}+p_{w}\right)+\bar{c} U_{k+1}\left(p_{\bar{s}}+p_{\bar{w}}\right) \tag{3.66}
\end{equation*}
$$

(3.53) and applying (3.66) twice let us obtain

$$
\begin{align*}
u u^{\prime} u u^{\prime} & =u u^{\prime} p_{r} U_{k+1} U_{k+2} p_{[t]} U_{k+2} U_{k+1} p_{r} \\
& =\left([2] c U_{k+1}\left(p_{s}+p_{w}\right)+[2] \bar{c} U_{k+1}\left(p_{\bar{s}}+p_{\bar{w}}\right)\right) U_{k+2} p_{[t]} U_{k+2} U_{k+1} p_{r} \\
& =[2]\left(c\left(p_{s}+p_{w}\right)+\bar{c}\left(p_{\bar{s}}+p_{\bar{w}}\right)\right) p_{r} U_{k+1} U_{k+2} p_{[t]} U_{k+2} U_{k+1} p_{r} \\
& =[2]\left(c\left(p_{s}+p_{w}\right)+\bar{c}\left(p_{\bar{s}}+p_{\bar{w}}\right)\right) u u^{\prime} \\
& =[2]\left(c\left(p_{s}+p_{w}\right)+\bar{c}\left(p_{\bar{s}}+p_{\bar{w}}\right)\right)\left(c U_{k+1}\left(p_{s}+p_{w}\right)+\bar{c} U_{k+1}\left(p_{\bar{s}}+p_{\bar{w}}\right)\right) \\
& =[2]\left(c\left(p_{s}+p_{w}\right)+[2] \bar{c}\left(p_{\bar{s}}+p_{\bar{w}}\right)\right)^{2} U_{k+1} \\
& =[2]\left(c^{2}\left(p_{s}+p_{w}\right)+[2] \bar{c}^{2}\left(p_{\bar{s}}+p_{\bar{w}}\right)\right) U_{k+1} \tag{3.67}
\end{align*}
$$

where we also used the second part of Remark 3.1.8 for $r$ and its extensions $s, w$, $\bar{s}$ and $\bar{w}$. With $u u^{\prime} p_{t}=0=u u^{\prime} p_{\bar{t}}$, because of $U_{k+1} p_{t}=0=U_{k+1} p_{\bar{t}}$, (3.67) implies now

$$
\begin{align*}
u u^{\prime} u u^{\prime}-[2](c+\bar{c}) u u^{\prime} & =-[2] c \bar{c}\left(p_{s}+p_{w}\right) U_{k+1}-[2] \bar{c} c\left(p_{\bar{s}}+p_{\bar{w}}\right) U_{k+1} \\
& =-[2] c \bar{c} p_{r} U_{k+1} . \tag{3.68}
\end{align*}
$$

Dividing (3.68) by [2]cc gives (3.55), which was to prove.

Finally setting

$$
\hat{u}=-\frac{u u^{\prime} u-[2](c+\bar{c}) u}{[2] c \bar{c}}
$$

implements with (3.54) and (3.55) the wanted equivalence, namely we obtain that

$$
u^{\prime} \hat{u}=[2] p_{[t]} \text { and } \hat{u} u^{\prime}=p_{r} U_{k+1}
$$

Note $u^{\prime}$ and $\hat{u}$ are evaluable since $p_{[t]}$ and $p_{r} U_{k+1}$ are, which we mentioned in the beginning of this proof.

The following lemma is only needed for a later proof. Recall that $[n]_{q}$ denotes the quantum evaluated at $q$.

Lemma 3.2.15. For $m \geq 1$ the function

$$
\begin{equation*}
T_{k}=\left(\frac{[m l-2]}{[m l-1]}\right)^{k-1} \frac{[m l-1]}{[m l]}+\left(\frac{[m l+2]}{[m l+1]}\right)^{k-1} \frac{[m l+1]}{[m l]} \tag{3.69}
\end{equation*}
$$

is evaluable at $q$ and $T_{k}(q)=\left([2]_{q}^{2}+2-2 k\right)[2]_{q}^{k-2}$. In particular, $T_{k}(q) \neq 0$ for $k \geq 3$.

Proof. It is sufficient to treat the case $m=1$, since a $2 l$ th root of unity is also a $m l$ th root of unity. We proceed by induction over $k$. For $k=1$, it is clear that $T_{k}=$ [2] and $T_{k}(q)=[2]_{q}$. For bigger $k$ the function $T_{k}$ satisfies

$$
\begin{align*}
T_{k}= & \left(\frac{[l-2]}{[l-1]}\right)^{k-1} \frac{[l-1]}{[l]}+\left(\frac{[l+2]}{[l+1]}\right)^{k-1} \frac{[l+1]}{[l]} \\
= & \frac{[l-2]}{[l-1]}\left(\left(\frac{[l-2]}{[l-1]}\right)^{k-2} \frac{[l-1]}{[l]}+\left(\frac{[l+2]}{[l+1]}\right)^{k-2} \frac{[l+1]}{[l]}\right) \\
& +\left(\frac{[l+2]}{[l+1]}\right)^{k-2}\left(\frac{[l+2]}{[l+1]} \frac{[l+1]}{[l]}-\frac{[l-2]}{[l-1]} \frac{[l+1]}{[l]}\right) \\
= & \frac{[l-2]}{[l-1]} T_{k-1}+\left(\frac{[l+2]}{[l+1]}\right)^{k-2}\left(\frac{[l+2]}{[l]}-\frac{[l-2]}{[l-1]} \frac{[l+1]}{[l]}\right) . \tag{3.70}
\end{align*}
$$

A little calculation, which is left to the reader, shows that

$$
\frac{[l+2]}{[l]}-\frac{[l-2]}{[l-1]} \frac{[l+1]}{[l]}=\frac{\left(v^{l}+v^{-l}\right)\left(v-v^{-1}\right)}{v^{l-1}-v^{-l+1}}
$$

which is evaluable at $q$ with value $\frac{(1+1)\left(q-q^{-1}\right)}{q^{-1}-q}=-2$. Therefore, (3.70) and the induction hypothesis imply that

$$
T_{k}=\frac{[l-2]}{[l-1]} T_{k-1}+\left(\frac{[l+2]}{[l+1]}\right)^{k-2}\left(\frac{[l+2]}{[l]}-\frac{[l-2]}{[l-1]} \frac{[l+1]}{[l]}\right)
$$

is evaluable in $q$ with value $T_{k}(q)=\left([2]_{q}^{2}+2-2 k\right)[2]_{q}^{k-2}$.

The next lemma (cf. (GW93, Big Diamond Lemma)) is somehow the peak of this section, it will provide the means to show the main result of this section, namely the statement regarding evaluability of regular critical tableaux. Moreover, it will be necessary to discuss nilpotent elements in the next section. To fix termininology for what comes, an idempotent $p \in T L_{n}$ is said to dominate an idempotent $f \in T L_{n}$, if $p f=f=f p$.

Lemma 3.2.16 (Big Diamond Lemma). Let be a tableau such that both $t$ and its critical subtableau $r$ end on the same critical line. If $p_{r}$ is evaluable, then $p_{[t]}$ dominates an evaluable minimal idempotent $f$, such that the coefficients $a$ and $b$ in $p_{t} f p_{t}=a p_{t}$ and $p_{\bar{t}} f p_{\bar{t}}=b p_{\bar{t}}$ have simple poles at $v=q$. Moreover, $f$ satisfies $f p_{t} f=a f$ and $f p_{\bar{t}} f=b f$.

Proof. Suppose that $\operatorname{Shape}(r)=\mu \in \operatorname{Par}_{2}(k)$ is on the $m$ th critical line and that Shape $(t)=\left(\mu_{1}+n, \mu_{2}+n\right)$. Instead of considering $t$ it is more convenient to first treat the following "zig-zag" path

$$
s=s_{n}=\left(r \rightarrow \mu^{(k+1)} \rightarrow \cdots \rightarrow \mu^{(k+2 n)}\right)=\left(\mu_{1}+n, \mu_{2}+n\right)
$$

defined by

$$
\epsilon_{i}\left(s_{n}\right)=\left\{\begin{array}{lll}
\epsilon_{i}(r), & \text { if } i \leq k, \\
-1, & \text { if } i=k+1, \\
-1, & \text { if } i=k+2 j, & \text { for } 1 \leq j \leq n-1, \\
+1, & \text { if } i=k+2 j+1, & \text { for } 0 \leq j \leq n-1, \\
+1, & \text { if } i=k+2 n &
\end{array}\right.
$$

Figure 12 shows the paths $s$ and $\bar{s}$ and their, supposed that $p_{r}=f_{r} x p_{d} \tilde{x}$. To understand the pictures, the following remark might be helpful:

1. For each pair $(k+2 j+1, k+2 j+2)$, the "zig-zag" path $s$ has a "hook", namely $\epsilon_{k+2 j+1}(s)=+1$ and that $\epsilon_{k+2 j+2}=-1$. These "hooks" correspond to the $n-2$ Jones-Wenzl projectors $p_{d-1}$ sitting above and below the middle projector $p_{d}$ in $p_{s}$.
2. $\bar{s}$ has one "hook" more, and these to the $n-1$ Jones-Wenzl projectors $p_{d+2}$ sitting above or below the middle.

Moreover following carefully the construction of the coefficients $f_{s}$ and $f_{\bar{s}}$ for $p_{s}$ and $p_{\bar{s}}$ in Definition 3.1.5 shows that

$$
\begin{align*}
& f_{s}=f_{r} \cdot\left(\frac{[d-1]}{[d]}\right)^{n-1} \frac{[d]}{[d+1]},  \tag{3.71}\\
& f_{\bar{s}}=f_{r} \cdot\left(\frac{[d+2]}{[d+3]}\right)^{n-1} \frac{[d+1]}{[d+2]} . \tag{3.72}
\end{align*}
$$



Figure 12: A path $s$, its conjugate $\bar{s}$ and their path idempotents $p_{s}$ and $p_{\bar{s}}$.

The elements $u, u^{\prime}, w$ and $w^{\prime}$ are now defined by

$$
\begin{aligned}
& u=u_{n}=\left(U_{k+1} U_{k+2} \cdots U_{k+2 n-1}\right)\left(U_{k+2} U_{k+4} \cdots U_{k+2 n-2}\right) \text { and } w=u p_{[s]} \text {, } \\
& u^{\prime}=u_{n}^{\prime}=\left(U_{k+2} U_{k+4} \cdots U_{k+2 n-1}\right)\left(U_{k+1} U_{k+2} \cdots U_{k+2 n-1}\right) \text { and } w=p_{[s]} u^{\prime} .
\end{aligned}
$$

The outline of this proof is now to show that, if rescaled by an non-zero evaluable scalar, the element $w w^{\prime}$ enjoys the properties of the idempotent $f$ of the statement.

1. We first show that $w^{\prime}$ and $w$ are evaluable. Let $s_{2}, \ldots, s_{n-2}$ denote the "zigzag" paths ending in $\left(\mu_{1}+2, \mu_{2}+2\right), \ldots,\left(\mu_{1}+n-1, \mu_{2}+n-1\right)$ and let $s_{1}$ denote the path ending in $\left(\mu_{1}+1, \mu_{2}+1\right)$ as in Lemma 3.2.12. Then $p_{\left[s_{1}\right]}$ is evaluable by Lemma 3.2.12 and $u_{i} p_{\left[s_{i}\right]}$ is evaluable by induction hypothesis for $2 \leq i \leq n-1$. However, every path idempotent $p_{\sigma}$, where $\sigma$ extends $r$, is either mapped to 0 under $U_{k+2} \cdots U_{k+2 n-2}$ by (3.7) or contained in the decomposition of one of the $p_{\left[s_{i}\right]}$ or of $p_{[s]}$, which implies

$$
u p_{r}=u p_{[s]}+\sum_{i=1}^{n-1} u p_{\left[s_{i}\right]}
$$

Thus $w$ is evaluable. Similarly, $w^{\prime}$ is too.
2. It is also clear that $w^{\prime} w$ is dominated by $p_{[s]}$.
3. The next step is to show the equations

$$
\begin{align*}
& p_{s} w^{\prime} w p_{s}=A p_{s}  \tag{3.73}\\
& p_{\bar{s}} w^{\prime} w p_{\bar{s}}=B p_{\bar{s}} \tag{3.74}
\end{align*}
$$

where $A$ and $B$ are coefficients having poles at $v=q$. Since
$p_{s} u^{\prime} u p_{s}$ equals

which implies

$$
\begin{align*}
& \text { ies }  \tag{3.75}\\
& p_{s} w^{\prime} w p_{s}=p_{s} p_{[s]} u^{\prime} u p_{[s]} p_{s}=p_{s} u^{\prime} u p_{s}=[2]^{n} \frac{f_{s}}{f_{r}} p_{s} .
\end{align*}
$$

Moreover, substituting (3.71) into (3.75) gives
$p_{s} w^{\prime} w p_{s}=[2]^{n}\left(\frac{[d-1]}{[d]}\right)^{n-1} \frac{[d]}{[d+1]} p_{s}=[2]^{n}\left(\frac{[m l-2]}{[m l-1]}\right)^{n-1} \frac{[m l-1]}{[m l]} p_{s}=: A_{n} p_{s}$
which is just (3.73). Now for the second equation, using

$$
\stackrel{k}{k+1}=\left([2]-\frac{[k]}{[k+1]}\right) \stackrel{1}{\uparrow}=\frac{[k+2]}{[k+1]} \frac{k}{\uparrow},
$$

we can calculate that $p_{\bar{s}} u^{\prime} u p_{\bar{s}}$ equals

which implies

$$
\begin{equation*}
p_{\bar{s}} u^{\prime} u p_{\bar{s}}=\frac{f_{\bar{s}}}{f_{r}}\left(\frac{[d+3]}{[d+2]}\right)^{2 n-2}[2]^{n}\left(\frac{[d+2]}{[d+1]}\right)^{2} p_{\bar{s}} \tag{3.76}
\end{equation*}
$$

But substituting (3.72) into (3.76) yields

$$
p_{\bar{s}} u^{\prime} u p_{\bar{s}}=\left(\frac{[d+3]}{[d+2]}\right)^{n-1}[2]^{n} \frac{[d+2]}{[d+1]} p_{\bar{s}}=[2]^{n}\left(\frac{[m l+2]}{[m l+1]}\right)^{n-1} \frac{[m l+1]}{[m l]} p_{\bar{s}},
$$

and moreover,

$$
p_{\bar{s}} w^{\prime} w p_{\bar{s}}=p_{\bar{s}} u^{\prime} u p_{\bar{s}}=[2]^{n}\left(\frac{[m l+2]}{[m l+1]}\right)^{n-1} \frac{[m l+1]}{[m l]} p_{\bar{s}}:=B p_{\bar{s}}
$$

which shows (3.74).
4. What is left is rescaleing $w^{\prime} w$ by a non-zero and evaluable coefficent, which turns it into an idempotent. Similar as before, one can calculate

$$
u^{\prime} u p_{s} u^{\prime} u p_{r}=u^{\prime} u p_{r} \cdot A \text { and } u^{\prime} u p_{\bar{s}} u^{\prime} u p_{r}=u^{\prime} u p_{r} \cdot B,
$$

which is leaft to the reader. This implies

$$
p_{[s]} u^{\prime} u p_{s} u^{\prime} u p_{[s]}=A w^{\prime} w \text { and } p_{[s]} u^{\prime} u p_{\bar{s}} u^{\prime} u p_{[s]}=B w^{\prime} w
$$

and moreover,

$$
w^{\prime} w w^{\prime} w=p_{[s]} u^{\prime} u p_{s} u^{\prime} u p_{[s]}+p_{[s]} u^{\prime} u p_{\bar{s}} u^{\prime} u p_{[s]}=(A+B) w^{\prime} w .
$$

On the other hand, since $A+B=T_{n}$ from (3.69), which is non-zero and evaluable, setting $f=\frac{1}{T_{n}} w^{\prime} w$ provides an idempotent enjoying the properties of the statement for the idempotent $p_{[s]}$.

It remains to observe, that the equivalence given in Lemma 3.2.13 preserves poles, so one can easily deduce the claim of the Big Diamond Lemma for general $t$ from that for the chosen $s$.

So far we did not construct that many evaluable path idempotents, most of our proven statements assumed the existence of such an evaluable path idempotent. But now we are in a good shape to obtain family of evaluable path idempotents:

## Proposition 3.2.17.

1. Every regular critical tableau is evaluable.
2. Two regular critical tableaux with same critical subtableau and same shape are equivalent.


Figure 13: A regular tableau $t, s=s_{j}(t)$ its conjugate $\bar{s}$.
Proof. We first show that every regular critical tableau $t$ is evaluable by using induction over the number of critical partitions on $t$ considered as a path. If this number is 1 or 2, than $t$ is evaluable by Proposition 3.2.10 respectively Lemma 3.2.12.

Therefore $t$ is assumed to have at least 3 critical partitions and induction hypothesis assures every regular tableau with fewer critical partitions is evaluable.

1. If $\bar{t}$ does not exist, we know that $t$ ends on the second critical line and its critical subtableau on the first. In particular, Lemma 3.2.13 and the induction hypothesis imply that $p_{[t]}=p_{t}$ is evaluable.
2. Now we consider the case where $t=\left([1] \rightarrow \lambda^{(2)} \rightarrow \cdots \rightarrow \lambda^{(k)}=\lambda\right.$ ) has its last three critical partitions $\lambda^{(i)}, \lambda^{(j)}$ and $\lambda^{(k)}=\lambda$ on the $m$ th, $(m \pm 1)$ th and on the $m$ th critical lines. By excluding the first case, we can assume that $p_{\bar{t}}$ exists. Set $s=s_{j}(t)$. The situation is illustrated in Figure 13. Now the subpath of $t$ of shape $\lambda^{(j)}$ is evaluable by induction so by Lemma 3.2.13 also $p_{[t]}$ is evaluable. But also the subpath ending on $\lambda^{(i)}$ is evaluable by induction hence $p_{[s]}$ is. Therefore both $p_{[s]}$ and $p_{[t]}$ are evaluable. By Lemma 3.2.16, the idempotent $p_{[s]}$ dominates an evaluable minimal idempotent $f$, such that

$$
\begin{equation*}
p_{s} f p_{s}=b p_{s} \text { and } f p_{s} f=b f \tag{3.77}
\end{equation*}
$$

where $b$ has a simple pole in $q$. Defining $u=f U_{j} p_{[t]}$ and $u^{*}=p_{[t]} U_{j} f$ and using (3.7) implies first

$$
\begin{equation*}
p_{\bar{t}} U_{j}=0=U_{j} p_{\bar{t}} \tag{3.78}
\end{equation*}
$$

and second with Corollary 3.1.19 that

$$
\begin{equation*}
p_{t} U_{j} p_{\bar{s}}=0=p_{\bar{s}} U_{j} p_{t} \tag{3.79}
\end{equation*}
$$

Furthermore, the equations (3.77), (3.78) and (3.79) together assure that

$$
\begin{align*}
u^{*} u & =p_{[t]} U_{j} f U_{j} p_{[t]}=p_{t} U_{j} f U_{j} p_{t}=p_{t} U_{j} p_{[s]} f p_{[s]} U_{j} p_{t} \\
& =p_{t} U_{j} p_{s} f p_{s} U_{j} p_{t}=p_{t} U_{j} b p_{s} U_{j} p_{t} \tag{3.80}
\end{align*}
$$

holds. In the proof of Proposition 3.2.10, we saw that

$$
\begin{equation*}
p_{t} U_{j} p_{s} U_{j} p_{t}=c p_{t} \text { where } c=\frac{[d+2][d]}{[d+1]^{2}} \text { and } d+2=(m \pm 1) l \tag{3.81}
\end{equation*}
$$

so substituting (3.81) in (3.80) implies

$$
\begin{equation*}
p_{t}=\frac{1}{c b} u^{*} u \tag{3.82}
\end{equation*}
$$

which is then evaluable, since $b$ has a simple pole in $q$ and $c$ a simple zero. On the other side, (3.77), (3.78) and (3.79) also ensure

$$
\begin{align*}
u u^{*} & =f U_{j} p_{[t]} U_{j} f=f p_{s} U_{j} p_{t} U_{j} p_{s} f+f \underbrace{p_{\bar{s}} U_{j} p_{\bar{t}} U_{j} p_{\bar{s}}}_{=0} f  \tag{3.83}\\
& =f c p_{s} f=b c f
\end{align*}
$$

where we used $p_{s} U_{j} p_{t} U_{j} p_{s}=c p_{s}$ similar to (3.81). However, (3.82) and (3.83) imply that

$$
\begin{equation*}
f=\frac{1}{b c} u u^{*} \text { and } p_{t}=\frac{1}{b c} u u^{*} \tag{3.84}
\end{equation*}
$$

are equivalent.
3. If the last three critical diagrams on $t$ are all on different critical lines, then $\bar{t}$ is evaluable by the previous case. Since $p_{[t]}$ is evaluable by the induction hypothesis and Lemma 3.2.13, also $p_{t}$ is evaluable.

Any other critical tableau $s$ with same critical subtableau can be obtained by a sequence of simple transpositions. The same proof as in Proposition 3.2.10 shows, that $p_{s}$ and $p_{t}$ are equivalent, since the equivalence in Proposition 3.2.10 preserves poles and zeros.

Now that regular critical tableaux of same shape and same critical subtableaux are equivalent, this section ends with the following lemma implementing an equivalence between two certain regular critical tableaux of same shape, but having different critical subtableaux.

Lemma 3.2.18. Fix $m \geq 1$ and let be the unique tableau of shape $\lambda=((m+1) l-$ $1, l)$ which passes through the diagram $((m+1) l-1)$, i.e. $t$ corresponds to the path obtained by going $(m+1) l-1$ steps to the right followed by $l$ steps to the left in the branching graph (see Figure 14). Then $p_{t}$ is equivalent to $p_{r}$, where $r$ is the tableau of shape $\lambda$ with critical subtableau of shape $((m-1) l-1)$.


Figure 14: A regular tableau $t, s=s_{j}(t)$, its conjugate $\bar{s}$ and $r=s_{i}(\bar{s})$.

Proof. Set $s=s_{i}(t)$ and $r=s_{j}(\bar{s})$, where $i=(m+1) l-1$ and $j=m l-1$. Now $p_{t}, p_{r}$ and $p_{\bar{r}}$ are evaluable by Proposition 3.2.17, in fact, (3.84) implies

$$
p_{t}=\frac{1}{b c} u^{*} u \text { and } f=\frac{1}{b c} u u^{*}
$$

where $c=\frac{[(m+1) l][(m+1) l-2]}{[(m+1) l-1]^{2}}$ is given by (3.81) and where $f$ is an evaluable idempotent satisfying

$$
p_{s} f p_{s}=b p_{s} \text { and } p_{\bar{s}} f p_{\bar{s}}=\bar{b} p_{\bar{s}}
$$

With $w=f U_{j} p_{[r]}$ and $w^{*}=p_{[r]} U_{j} f$, one can show, similar as for (3.84), that

$$
p_{r}=\frac{1}{\bar{b} \bar{c}} w^{*} w \text { and } f=\frac{1}{\bar{b} \bar{c}} w w^{*}
$$

where $\bar{c}=\frac{[(m-1) l+2][(m-1) l]}{[(m-1) l+1]^{2}}$. But this actually means, that $p_{r}$ and $f$ are equivalent, yielding equivalence between $p_{r}$ and $p_{t}$.

Now that we have proven most of the technical properties concering evaluability and equivalence between path idempotents and other idempotents, we can turn our attention to some more interesting results.

Section 3.2.2 The structure of $T L_{n}(q)$ at a root of unity
This section is dedicated to structure results in the non-semisimple TemperleyLieb algebra. As before, the statements of this section can be found in (GW93,

Section 2). The presented proofs are basically those of (GW93), however, we filled in some details. First, some new termininology is needed:

Definition 3.2.19. Let $\lambda$ be a diagram between the $m$ th and the $m+1$ st critical line, i.e. $m l<w(\lambda)<(m+1) l$, where $m=0$ is allowed.

1. We say that $t \in \operatorname{Std}(\lambda)$ is coming from the right, if its critical tableau ends on the $m+1$ th critical line, otherwise it is said to come from the left. $R(\lambda) \subset$ $\operatorname{Std}(\lambda)$ will denote the subset of tableaux coming from the right and $L(\lambda)$ the subset of those coming from the left.
2. The elements $z_{\lambda}^{R}$ and $z_{\lambda}^{L}$ are defined by

$$
z_{\lambda}^{R}=\sum_{t \in R(\lambda)} p_{[t]} \text { and } z_{\lambda}^{L}=\sum_{t \in L(\lambda)} p_{[t]} .
$$

with $f_{\lambda}:=\# \operatorname{Std}(\lambda), f_{\lambda}^{L}:=\# L(\lambda)$ and $f_{\lambda}^{R}:=\# R(\lambda)$.
Before formulating the main result of this section, we make the following observation:

Remark 3.2.20. 1. Clearly $\operatorname{Std}(\lambda)$ is the disjoint union of $L(\lambda)$ and $R(\lambda)$.
2. The summands of $z_{\lambda}^{L}$ and $z_{\lambda}^{R}$ are not necessarily evaluable.
3. Let $A_{1}^{(1)}$ be the group of reflections on $\mathbb{Z}$ about the number $m l, m \in \mathbb{Z}$, which acts on diagrams of fixed size by reflecting about critical lines. In particular, if $\mu$ is the diagram obtained from $\lambda$ by reflecting about the $m+1$ st critical line, we see that $z_{\lambda}^{R}=z_{\mu}^{L}$ and moreover $f_{\lambda}^{R}=f_{\mu}^{L}$. Consequently one obtains $f_{\lambda}^{L}=f_{\lambda}-f_{\lambda}^{R}=f_{\lambda}-f_{\mu}^{L}$ and furthermore $f_{\lambda}^{L}=f_{\lambda}-f_{\mu}+\ldots$.
[ $\lambda$ ] denotes be the orbit of $\lambda$ under the action of $A_{1}^{(1)}$ and $z_{[\lambda]}$ is defined to be $\sum_{\mu \in[\lambda]} z_{\mu}$. Moreover, $\mu$ and $v$ in [ $\left.\lambda\right]$ are said to be adjacent, if there is exactly one critical line between $\mu$ and $v$ and they are obtained from each other by reflecting about that line. By the radical, we mean the Jacobson radical, i.e. the ideal generated by all elements annihilating all modules. For example all nilpotent elements are included in the radical. We denote it by rad. The main result of this section is the following statement, which can be found in (GW93, Theorem 2.3):

Theorem 3.2.21. If $\lambda$ is non-critical, then $z_{[\lambda]}(q)$ is a minimal central idempotent in $T L_{n}(q)$. The radical of $z_{[\lambda]} T L_{n}(q)$ is nilpotent of order 3 and is spanned by the spaces $z_{\mu}^{L} T L_{n} z_{v}^{L}(q)$ for pairs of adjacent diagrams $\mu, v$ in the orbit $[\lambda]$ and by the algebras $\operatorname{rad}\left(z_{\mu}^{L} T L_{n} z_{\mu}^{L}(q)\right)$ for $\mu \in[\lambda]$. Moreover, the maximal semisimple quotient of $z_{[\lambda]} T L_{n}(q)$ is isomorphic to $\bigoplus_{\mu \in[\lambda]} M_{f_{\mu}} \mathbb{C}$.

To investigate this, more notation is needed:

Definition 3.2.22. Let $\lambda=\left(\lambda_{1}, \lambda_{2}\right) \in \operatorname{Par}_{2}(n), n \geq 2$ be a partition. If $\lambda_{1}>\lambda_{2}, \alpha=$ $\left(\lambda_{1}-1, \lambda_{2}\right)$ is called the left subdiagram and if $\lambda_{2} \geq 1$, the partition $\beta=\left(\lambda_{1}, \lambda_{2}-1\right)$ is called the right subdiagram.

Remark 3.2.23. It is clear that $f_{\lambda}=f_{\alpha}+f_{\beta}$ and moreover, the folllowing identities hold:

$$
\begin{array}{ll}
f_{\lambda}=f_{\alpha}^{L}+f_{\beta}^{L} \text { and } f_{\lambda}^{R}=f_{\alpha}^{R}+f_{\beta}^{R}, & \text { if } \alpha, \beta \text { are non-critical, } \\
f_{\lambda}^{L}=f_{\alpha}+f_{\beta}^{L} \text { and } f_{\lambda}^{R}=f_{\beta}^{R}, & \text { if } \alpha \text { is critical, } \\
f_{\lambda}^{R}=f_{\beta}+f_{\alpha}^{R} \text { and } f_{\lambda}^{L}=f_{\alpha}^{L}, & \text { if } \beta \text { is critical. } \tag{3.87}
\end{array}
$$

Together with $f_{\lambda}=f_{\lambda}^{L}+f_{\lambda}^{R}$, these formulas will allows us to use induction in what follows.

To simplify the proof of the next proposition, it is convenient to refine the notation of matrix units, see Remark 3.2.7). Therefore we fix the following termininology:

Definition 3.2.24. Let $z \in T L_{n}$ be an idempotent and $f$ a natural number. $A z-$ system of matrix units $U=\left\{u_{i j}, 1 \leq i, j \leq f\right\}$ is a system of matrix units $u_{i j} \neq 0$ in $T L_{n}$, such that the idempotents $u_{i i}$ are pairwise orthogonal and dominated by $z$ for all $i$ and such that they sum up to $z . U$ is said to have order $f . U$ contains an idempotent $p$, if $p=u_{i i}$ for an $i \in\{1, \ldots, f\}$. Moreover, $U$ is called minimal, if $f$ is the maximal natural number such that there exists a $z$-system of evaluable matrix units of order $f$. Finally, $U$ is said to be equivalent to an idempotent $r$ if one (or all) elements of $U$ are equivalent to $r$.

Remark 3.2.25. 1. Let $U=\left\{u_{i j}\right\}$ be a $z$-system containing $p$ and assume that $p$ is equivalent to $p^{\prime}$, i.e. $u u^{*}=p^{\prime}$ and $u^{*} u=p$. Then there exists a $z$-system $V=\left\{v_{i j}\right\}$ containing $p^{\prime}$ of same order than $U$, by setting

$$
v_{i j}= \begin{cases}u u_{1 j}, & i=1 \\ u_{i 1} u^{*}, & j=1 \\ u_{i j}, & \text { otherwise }\end{cases}
$$

where we assumed without loss of generality that $p=u_{11}$.
2. To obtain a $z_{\lambda}$-system of evaluable matrix units of order $f$, clearly it suffices by Remark 3.2.7 to have $f$ pairwise orthogonal and pairwise equivalent evaluable idempotents dominated by $z_{\lambda}$.

Now having a bunch of statements coming from Section 3.2.1 and also having fixed the termininology, following (GW93, Proposition 2.1), we can produce a sufficient number of evaluable idempotents.

## Proposition 3.2.26.

1. If $\lambda$ is a critical partition, then there exists a minimal $z_{\lambda}$-system of evaluable matrix units of order $f_{\lambda}$ containing at least one $p_{t}$ for $t \in \operatorname{Std}(\lambda)$ regular. Moreover for any two regular tableaux $s$ and $t$ in $\operatorname{Std}(\lambda)$ the idempotents $p_{t}$ and $p_{s}$ are equivalent.
2. If $\lambda$ is a non-critical diagram, then there exists a minimal $z_{\lambda}^{L}$-system of evaluable matrix units or order $f_{\lambda}^{L}$ containing at least one $p_{[t]}$ for $t \in L(\lambda)$ regular. Furthermore, for any two regular tableaux $s$ and $t$ in $L(\lambda)$ the elements $p_{[t]}$ and $p_{[s]}$ are equivalent.

The proof takes a couple of pages. However, since the statement is proved by one induction, we believe, that it is better to understand, if not splitted up into smaller pieces. The reader may forgive us this inconvenience.

Proof. The statements of the proposition hold already in the following cases:

1. If $\lambda$ is critical with only one row, i.e. $\lambda_{2}=0$, then clearly $z_{\lambda}=p_{t}$. Now the statements are just a reformulation of Proposition 3.2.17.
2. If $\lambda$ is non-critical and has only one row, then its critical subdiagram is evaluable by Proposition 3.2.17 and by Lemma 3.2.13 also $z_{\lambda}^{L}=p_{[t]}$ is then evaluable. Likewise equivalence follows by Lemma 3.2.13.
3. If $\lambda$ is a diagram left of the first critical line, i.e. if $w(\lambda)<l$, then Proposition 3.2.10 implies that $z_{\lambda}=\sum_{t \in \operatorname{Std}(\lambda)} p_{t}$ is evaluable and it also states that all the summands are equivalent.

We proceed by induction on $\lambda_{1}+\lambda_{2}$ and by excluding the above three cases, we can assume that $\lambda_{2} \neq 0$ and that the statements of the proposition hold for all diagrams of size less then $\lambda_{1}+\lambda_{2}$. In particular, both $\alpha$ and $\beta$ exist and they enjoy the statements of the proposition.

Moreover, $z_{\lambda}$ has a minimal system of matrix units of order $f_{\lambda}$ containing a regular $p_{t}$, but which is not necessarily evaluable. Therefore it suffices to find a system of evaluable matrix units of order $f_{\lambda}$ containing a regular $p_{t}$, it will automatically be minimal. The same is of course true for $z_{\lambda}^{L}$. There are four cases to consider, namely the cases where

1. $\lambda$ is critical,
2. $\alpha$ is critical,
3. $\beta$ is critical or
4. neither $\lambda$, nor $\alpha$ and nor $\beta$ are critical.

We start be regarding the first one:

1. If $\lambda$ is on the $m$ th critical line, let $\mu=\left(\lambda_{1}-1, \lambda_{2}-1\right)$. Let $r \in \operatorname{Std}(\mu)$ be a tableau, such that its critical subtableau is of shape $((m-1) l-1)$ or if $m=1$, let $r$ have no critical subtableau. Since $r$ is regular, it is evaluable by Proposition 3.2.10, if $m=1$, and by Proposition 3.2.17, if $m \neq 1$. Let $t \in \operatorname{Std}(\lambda)$, such that $t$ and $r$ enjoy the assumptions of Lemma 3.2.12, the situation is then described by the first picture in Figure 9. In particular, $t$ is also regular, thus evaluable, similar as $r$ is, by applying Proposition 3.2.10 respectively Proposition 3.2.17.
(a) We first produce a $z_{\alpha}^{L} z_{\lambda}$-system of evaluable matrix units of order $f_{\alpha}^{L}$ containing $p_{t}$. By induction hypothesis, there exists a minimal $z_{\alpha}^{L_{-}}$ system $U=\left\{u_{i j}\right\}_{1 \leq i, j \leq f_{\alpha}^{L}}$ of evaluable matrix of order $f_{\alpha}^{L}$ containing $p_{[w]}=u_{11}$ for some $w \in L(\alpha)$. However, by Proposition 3.2.10 (if $m=1$ ) or by Proposition 3.2.17 (if $m \neq 1$ ), we see that $p_{[w]}$ is equivalent to $p_{\left[t^{\prime}\right]}$, thus by Remark 3.2.25, we can assume that $w=t^{\prime}$.
Now set $v_{i j}=u_{i 1} p_{t} u_{1 j}$ for $1 \leq i, j \leq f_{\alpha}^{L}$. Then $V=\left\{v_{i j}\right\}$ is a $z_{\alpha}^{L} z_{\lambda^{-}}$ system of evaluable matrix units of order $f_{\alpha}^{L}$, which contains $p_{t}=v_{11}$.
(b) Now we produce a $z_{\beta}^{L} z_{\lambda}=z_{\alpha}^{R} z_{\lambda}$-system of evaluable matrix units of order $2 f_{\mu}=f_{\alpha}^{R}+f_{\beta}^{L}$, which is equivalent to $p_{t}$.
By induction assumption, there exists $z_{\mu}$-system $U$ of evaluable matrix units containing $p_{r}$ (using Remark 3.2.25) and which is of order $f_{\mu}$.
i. Every $x \in U$ is equivalent to $p_{r}$, which implies that also $x U_{n-1}$ and $x\left([2]-U_{n-1}\right) z_{\lambda}$ are equivalent to $p_{r} U_{n-1}$ and $p_{r}\left([2]-U_{n-1}\right) z_{\lambda}$, since $x$ and $p_{r}$ are elements of $T L_{n-2}$ and $U_{n-1}$ commutes with $T L_{n-2} \subset$ $T L_{n}$. In particular, this means that there are $2 f_{\mu}$ evaluable, pairwise orthogonal and pairwise equivalent idempotents all equivalent to $p_{r} U_{n-1}$.
ii. Since $t$ was chosen to satisfy the assumptions of Lemma 3.2.12 together with $r$, all those $2 f_{\mu}$ idempotents of the previous part are equivalent to [2] $p_{t}$. Consequently, dividing by [2] yields $2 f_{\mu}=$ $f_{\alpha}^{R}+f_{\beta}^{L}$ evaluable, pairwise orthogonal and pairwise equivalent idempotents all equivalent to $p_{t}$.
All these idempotents are dominated by $z_{\beta}^{L} z_{\lambda}=z_{\alpha}^{R} z_{\lambda}$, which means, that they give rise to a $z_{\alpha}^{R} z_{\lambda}$-system of evaluable matrix units of order $2 f_{\mu}=f_{\alpha}^{R}+f_{\beta}^{L}$, which is equivalent to $p_{t}$.
(c) Now we look for a $z_{\beta}^{R} z_{\lambda}$-system of evaluable matrix units of order $f_{\beta}^{R}$ equivalent to $p_{t}$.
i. If $\lambda_{2}<l$, then $f_{\beta}^{R}=0$, thus there would be nothing to show.
ii. If $\lambda_{2}=l$, then $f_{\beta}^{R}=1$ implies, that there is exactly one tableau $\hat{t}$ of shape $\lambda$ with critical subtableau of shape $((m+1) l+1)$. Moreover, $p_{\hat{t}}$ is evaluable by Proposition 3.2.17 and equivalent to a path
idempotent $p_{\hat{r}}$ by Lemma 3.2.18, which is then again equivalent to $p_{t}$ by Proposition 3.2.17, thus $p_{\hat{t}}$ is equivalent to $p_{t}$.
iii. Now suppose that $\lambda_{2}>l$. Let $\hat{r} \in \operatorname{Std}(\mu)$ with critical subtableau of shape $((m+1) l-1)$ and $\hat{t} \in \operatorname{Std}(\lambda)$ with same critical subtableau satisfy the assumptions of Lemma 3.2.12. The situation corresponds to the second picture in Figure 9.
By the induction assumption for $z_{\beta}^{R}$, which equals to $z_{\bar{\beta}}^{L}$, where $\bar{\beta}$ is the diagram obtained from $\beta$ by reflecting it about the $m+1$ th critical line, we can argue as in the first subcase (1.a) to obtain a $z_{\beta}^{R} z_{\lambda}$-system $V$ of evaluable matrix units of order $f_{\beta}^{R}$ containing $p_{\hat{t}}$. In particular, Lemma 3.2.12 implies that $V$ is equivalent to [2] ${ }^{-1} \cdot p_{\hat{r}} U_{n-1}$, which is equivalent to [2] ${ }^{-1} p_{r} U_{n-1}$ since $p_{\hat{r}}$ is equivalent to $p_{r}$ in $T L_{n-2}$ by induction assumption. But then $V$ is also equivalent to $p_{t}$.
Since $z_{\alpha}^{L} z_{\lambda}, z_{\alpha}^{R} z_{\lambda}$ and $z_{\beta}^{R} z_{\lambda}$ are orthogonal, we obtain by part (1.a), (1.b) and (1.c) a set of $f_{\alpha}+2 f_{\alpha}^{R}+f_{\beta}^{R}$ pairwise orthogonal and pairwise equivalent evaluable idempotents (all were equivalent to $p_{t}$ ). In particular, since

$$
\begin{equation*}
f_{\lambda}=f_{\alpha}+f_{\beta}=f_{\alpha}^{L}+f_{\alpha}^{R}+f_{\alpha}^{R}+f_{\beta}^{L}=f_{\alpha}^{L}+f_{\alpha}^{R}+f_{\alpha}^{R}+f_{\beta}^{R} \tag{3.88}
\end{equation*}
$$

this gives rise to a $z_{\lambda}$-system of evaluable matrix units containing $p_{t}$, where $t \in \operatorname{Std}(\lambda)$ is regular.
To finish this case, it remains to show, that for two regular tableaux $s$ and $t$ of shape $\lambda$ the elements $p_{t}$ and $p_{s}$ are equivalent. If $t$ and $s$ come both from the right or both from the left side, we know that $p_{\left[t^{\prime}\right]}$ and $p_{\left[s^{\prime}\right]}$ are equivalent by induction hypothesis (and evaluable since their critical subtableaux are regular). By using the fact that $z_{\lambda}$ is central and that $p_{s}=z_{\lambda} p_{\left[s^{\prime}\right]}$ and $p_{t}=z_{\lambda} p_{\left[t^{\prime}\right]}$ equivalence of $p_{t}$ and $p_{s}$ follows as well. On the other hand, we already showed the equivalence of two particular, regular tableaux, one coming from the left and one from the right, namely $p_{t}$ and $p_{\hat{t}}$.
2. Now suppose that $\lambda$ is non-critical and $\beta$ is critical and choose a regular $t \in L(\lambda)$. Induction hypothesis assures a $z_{\alpha}^{L}$-system $U=\left\{u_{i j}\right\}$ of evaluable matrix units of order $f_{\alpha}^{L}=f_{\lambda}^{L}$, such that $U$, using Remark 3.2.25, contains $p_{\left[t^{\prime}\right]}$. In particular, setting $v_{i j}=u_{i 1} p_{[t]} u_{1 j}$ and using $z_{\lambda}^{L}=z_{\alpha}^{L} z_{\lambda}$ yields a $z_{\lambda^{-}}^{L}$ system of evaluable matrix units of order $f_{\lambda}^{L}$ containing $p_{[t]}$. Escpecially $z_{\lambda}^{L}$ is evaluable. For two regular tableaux $t, s \in L(\lambda)$, the elements $p_{\left[t^{\prime}\right]}$ and $p_{\left[s^{\prime}\right]}$ are equivalent by induction assumption. But $z_{\lambda}^{L}$ being evaluable and commuting with $p_{\left[t^{\prime}\right]}$ and $p_{\left[s^{\prime}\right]}$ implies together with $p_{[s]}=p_{\left[s^{\prime}\right]} z_{\lambda}^{L}$ and $p_{[t]}=p_{\left[t^{\prime}\right]} z_{\lambda}^{L}$, that $p_{[s]}$ and $p_{[t]}$ are equivalent.
3. Now suppose that $\alpha$ is critical.
(a) We first produce a $z_{\lambda}^{L} z_{\alpha}$-system of evaluable matrix units of order $f_{\alpha}$ equivalent to $p_{[T]}$, such that $\bar{T} \in L(\lambda)$ is regular. Let $r$ be a critical
regular tableau ending in $\left(\alpha_{1}-1, \alpha_{2}-1\right)$ and let $t \in \operatorname{Std}(\alpha)$ enjoy together with $r$ the assumptions of Lemma 3.2.12. By induction, there is a $z_{\alpha}$-system $U$ of evaluable matrix units of order $f_{\alpha}$ containing a regular path idempotent $p_{w}$ and by using Remark 3.2.25, we can assume that $w=t$. But then by Lemma 3.2.12, $p_{t}$ is equivalent to $\frac{1}{[2]} p_{r} U_{n-1}$ and moreover, by Lemma 3.2.14 this is equivalent to $p_{[T]}$, where $\bar{T} \in L(\lambda)$ enjoys together with $r$ the assumptions of Lemma 3.2.14.
In particular, $U$ is a $z_{\lambda}^{L} z_{\alpha}$-system of evaluable matrix units of order $f_{\alpha}$ equivalent to $p_{[T]}$, where $\bar{T} \in L(\lambda)$ is regular.
(b) Induction hypothesis gives us a $z_{\beta}^{L}$-system $\left\{u_{i j}\right\}$ of evaluable matrix units of order $f_{\beta}^{L}$ containing a regular $p_{[w]}, w \in L(\beta)$ and Remark 3.2.25 ensures, that we can assume $w=T^{\prime}$. In particular, setting $v_{i j}=$ $u_{i 1} p_{[t]} u_{1 j}$ gives rise to a $z_{\beta}^{L} z_{\lambda}^{L}$-system of evaluable matrix units of order $f_{\beta}^{L}$ containing $p_{[T]}$.

By using $f_{\lambda}^{L}=f_{\alpha}+f_{\beta}^{L}$ and the fact that $z_{\beta}^{L} z_{\lambda}^{L}$ and $z_{\alpha} z_{\lambda}^{L}$ are orthogonal and sum up to $z_{\lambda}^{L}$, we obtain a $z_{\lambda}^{L}$-system of evaluable matrix units of order $f_{\lambda}^{L}$ containing $p_{[T]}$, where $\bar{T} \in L(\lambda)$ is regular. In particular, $z_{\lambda}^{L}$ is evaluable.
If $S$ and $T$ are two regular tableaux ending in $\lambda$ and if they are both coming from the left or the right, then $p_{[S]}$ and $p_{[T]}$ are equivalent due to the following facts:

$$
\begin{align*}
& p_{[T]}=z_{\lambda}^{L} p_{\left[T^{\prime}\right]}, \text { and } p_{[S]}=z_{\lambda}^{L} p_{\left[S^{\prime}\right]},  \tag{3.89}\\
& p_{\left[S^{\prime}\right]} \text { and } p_{\left[T^{\prime}\right]} \text { are equivalent by induction hypothesis, }  \tag{3.90}\\
& z_{\lambda}^{L} \text { is central in }\left(z_{\alpha}+z_{\beta}^{L}\right) T L_{n}\left(z_{\alpha}+z_{\beta}^{L}\right) . \tag{3.91}
\end{align*}
$$

But it is already known that $p_{[S]}=p_{S^{\prime}}$ and $p_{[T]}$ are equivalent for two particular, $S$ with $S^{\prime} \in \operatorname{Std}(\alpha)$ and $T$ with $T^{\prime} \in L(\beta)$, namely $S^{\prime}=\hat{t}$ and $T$ from above.
4. The last case to consider is that the diagrams $\lambda, \alpha$ and $\beta$ are all non-critical.

If $t \in L(\lambda)$ is any regular tableau, such that $t^{\prime} \in L(\alpha)$ and $t^{\prime \prime}$ has shape $\left(\lambda_{1}-1, \lambda_{2}-1\right)$, let $s=s_{n-1}(t) \in L(\lambda)$ such that $s^{\prime} \in L(\beta)$. Then $p_{\left[t^{\prime}\right]}, p_{[t]}, p_{[s]}$ and $p_{\left[s^{\prime}\right]}$ are evaluable by induction hypothesis and Lemma 3.2.13, since their common critical subtableau is regular and moreover, Lemma 3.2.13 implies that $p_{[t]}$ and $p_{[s]}$ are equivalent.
The induction hypothesis gives rise to a $z_{\alpha}^{L}$-system $U$ of evaluable matrix units of order $f_{\alpha}^{L}$, containing, using Remark 3.2.25, the idempotent $p_{\left[t^{\prime}\right]}$. Setting $v_{i j}=u_{i 1} p_{[t]} u_{1 j}$, yields as before a $z_{\alpha}^{L} z_{\lambda}^{L}$-system of evaluable matrix units of order $f_{\alpha}^{L}$ containing $p_{\left[t^{\prime}\right]}$. Argueing similarly implies existence of a $z_{\beta}^{L} z_{\lambda}^{L}$-system of evaluable matrix units of order $f_{\beta}^{L}$ containing $p_{[s]}$. Both


Figure 15: A non-critical tableau $t$ with critical subtableau of shape $(m l-1), s=$ $s_{m l-1}(t)$ and an extension $T$ of $t$.
systems form a $z_{\lambda}^{L}$ system of evaluable matrix units of order $f_{\lambda}^{L}=f_{\alpha}^{L}+f_{\beta}^{L}$ containing $p_{[t]}$, since $p_{[t]}$ and $p_{[s]}$ were equivalent.
Again, if $w$ is another regular path with $w \in L(\lambda)$, then $p_{[w]}$ is equivalent to $p_{[t]}$ due to (3.89),(3.90) and (3.91), where one has to replace $z_{\alpha}$ by $z_{\alpha}^{L}$ in (3.91).

Now all cases are treated.
With this family of evaluable idempotents at hand, we would like to describe the minimal central idempotents modulo the radical using the idempotents $z_{\lambda}$ respectively $z_{\lambda}^{L}$. This is formulated in the next theorem, however to prove it, we first need a lemma to identify nilpotent elements:

Lemma 3.2.27 ((GW93, Nilpotent elements)). Let t be a non-critical tableau with critical tableau of shape $(m l-1)$ for some $m \geq 1$ having its endpoint to the left of the mth critical line and moreover let the path s defined to be $s=s_{m l-1}(t)$, see also Figure 15. Defining $n_{[t]}=p_{[t]} U_{m l-1} p_{[t]}$ implies that:

1. $n_{[t]}$ is evaluable but nilpotent of order 2 , if evaluated at $v=q$.
2. $p_{[t]} T L_{n} p_{[t]}(q)$ is two dimensional and isomorphic to $\mathbb{C}[x] /\left(x^{2}\right)$.
3. $p_{[t]} T L_{n} p_{[s]}(q)$ and $p_{[s]} T L_{n} p_{[t]}(q)$ are one dimensional.
4. $p_{[t]} T L_{n} p_{[s]} T L_{n} p_{[t]}(q)=\mathbb{C} n_{[t]}(q)$ and

$$
p_{[s]} T L_{n} p_{[t]} T L_{n} p_{[s]}(q)= \begin{cases}\{0\}, & m=1 \\ \mathbb{C}_{[s]}(q), & \text { otherwise }\end{cases}
$$

5. $n_{[t]} T L_{n} p_{[s]}(q)=p_{[s]} T L_{n} n_{[t]}(q)=\{0\}$ and if $m>1$, also

$$
p_{[t]} T L_{n} n_{[s]}(q)=n_{[s]} T L_{n} p_{[t]}(q)=\{0\} .
$$

Proof. The critical subtableaux of $s$ and $t$ are regular hence evaluable and furthermore, by Lemma 3.2.13 $p_{[s]}$ and $p_{[t]}$ are evaluable.

1. (3.7) implies $U_{m l-1} p_{\bar{t}}=0$, which gives rise to

$$
\begin{equation*}
p_{[t]} U_{m l-1} p_{[t]}=p_{t} U_{m l-1} p_{t}=\frac{[m l]}{[m l-1]} p_{t} \tag{3.92}
\end{equation*}
$$

where the last equality follows by a little diagramatic argument. Note that $p_{t}$ is not necessary evaluable, thus evaluating at $q$ is not permitted to obtain 0 . But (3.92) implies

$$
\begin{equation*}
n_{[t]}^{2}=\frac{[m l]}{[m l-1]} p_{[t]} U_{m l-1} p_{[t]} \tag{3.93}
\end{equation*}
$$

which can be evaluated at $q$ yielding $n_{[t]}^{2}(q)=0$ since $[\mathrm{ml}]=0$. To see that $p_{[t]} U_{m l-1} p_{[t]}(q) \neq 0$, let $T$ be an extension of $t$ ending on the $m t$ critical line and also having its critical subtableau of shape $(m l-1)$, see Figure 15.
Applying Lemma 3.2.16 to $T$ implies the existence of an evaluable minimal idempotent $f$ dominated by $p_{[T]}$, such that the coefficient $b$ in $f p_{T} f=b f$ has a simple pole in $v=q$. (3.92) lets us then deduce

$$
\begin{equation*}
f p_{[t]} U_{m l-1} p_{[t]} f=\frac{[m l]}{[m l-1]} f p_{t} f=\frac{[m l]}{[m l-1]} f p_{T} f=\frac{[m l]}{[m l-1]} b f \tag{3.94}
\end{equation*}
$$

However, $\frac{[m l]}{[m l-1]} b$ is regular at $q$, thus we can evaluate (3.94) to obtain that $f p_{[t]} U_{m l-1} p_{[t]} f(q) \neq 0$. In particular, $n_{[t]}(q)=p_{[t]} U_{m l-1} p_{[t]}(q) \neq 0$.
2. $p_{t} U_{m l-1} p_{t} \neq 0$ in $T L_{n}$ implies

$$
\begin{equation*}
\operatorname{dim}_{\mathbb{C}(v)} p_{t} T L_{n} p_{t}=1=\operatorname{dim}_{\mathbb{C}(v)} p_{\bar{t}} T L_{n} p_{\bar{t}} \tag{3.95}
\end{equation*}
$$

and furthermore 3.1.19 gives rise to

$$
\begin{equation*}
\operatorname{dim}_{\mathbb{C}(v)} p_{\bar{t}} T L_{n} p_{t}=0=\operatorname{dim}_{\mathbb{C}(v)} p_{t} T L_{n} p_{\bar{t}} \tag{3.96}
\end{equation*}
$$

since $\bar{t}$ and $t$ are of different shapes. But then (3.95) and (3.96) together yield

$$
\operatorname{dim}_{\mathbb{C}(v)} p_{[t]} T L_{n} p_{[t]}=2
$$

which implies on the other hand with Lemma 3.2.3 that

$$
\begin{equation*}
\operatorname{dim}_{\mathbb{C}} p_{[t]} T L_{n} p_{[t]}(q)=2 \tag{3.97}
\end{equation*}
$$

Since $p_{[t]}$ and $n_{[t]}$ are non-zero, evaluable and since $p_{[t]}$ is idempotent, but $n_{[t]}$ nilpotent, they must form a basis. An isomorphism is given by mapping $p_{[t]}$ to 1 and $n_{[t]}$ to $x$.
3. This follows from applying Corollary 3.2.11, Corollary 3.1.19, (3.95) and Lemma 3.2.3.
4. The previous part implies

$$
\begin{equation*}
\operatorname{dim}_{\mathbb{C}} p_{[t]} T L_{n} p_{[s]} T L_{n} p_{[t]}(q) \leq 1 \tag{3.98}
\end{equation*}
$$

Setting

$$
\begin{equation*}
u=p_{[s]} U_{m l-1} p_{[t]} \text { and } u^{*}=p_{[t]} U_{m l-1} p_{[s]} \tag{3.99}
\end{equation*}
$$

lets us calculating

$$
\begin{align*}
u^{*} u & =p_{[t]} U_{m l-1} p_{[s]} U_{m l-1} p_{[t]}=p_{t} U_{m l-1} p_{s} U_{m l-1} p_{t} \\
& =\frac{[m l][m l-2]}{[m l-1][m l-1]} p_{t}=\frac{[m l-2]}{[m l-1]} n_{[t]}, \tag{3.100}
\end{align*}
$$

where the third equality follows by a little diagramatic argument, which was for example done in the proof of 3.2.10. Evaluating at $q$ implies that $u^{*} u(q)=[2]_{q} n_{[t]}(q)$, so $u(q)$ and $u^{*}(q)$ are both non-zero and forthermore $\operatorname{dim}_{\mathbb{C}} p_{[t]} T L_{n} p_{[s]} T L_{n} p_{[t]}(q)=1$.

- If $m=1$, then $p_{[s]}=p_{s}$ and $s$ is left to the first line, hence evaluable by Proposition 3.2.10. Moreover, one can obtain similar as (3.100) that

$$
\begin{equation*}
u u^{*}=p_{s} U_{m l-1} p_{t} U_{m l-1} p_{s}=\frac{[m l][m l-2]}{[m l-1][m l-1]} p_{s} \tag{3.101}
\end{equation*}
$$

Evaluating this at $v=q$ yields $u u^{*}(q)=0$, since $u$ and $u^{*}$ were evaluable. But $p_{[s]} T L_{n} p_{[t]} T L_{n} p_{[s]}$ is one dimensional and generated by $u u^{*}$, which is non-zero in $T L_{n}$, thus $p_{[s]} T L_{n} p_{[t]} T L_{n} p_{[s]}(q)=0$,

- On the other hand $m>1$ implies that the critical subtableau of $s$ is of shape $((m-1) l-1)$ and that $s_{(m-1) l-1}$ is admissible for $\bar{s}$. Defining

$$
\begin{equation*}
w=p_{[s]} U_{(m-1) l-1} p_{[t]} \text { and } w^{*}=p_{[t]} U_{(m-1) l-1} p_{[s]} \tag{3.102}
\end{equation*}
$$

allows us to obtain

$$
\begin{equation*}
w w^{*}=\frac{[(m-1) l][(m-1) l-2]}{[(m-1) l-1][(m-1) l-1]} p_{\bar{s}}=\frac{[(m-1) l-2]}{[(m-1) l-1]} n_{[s]}, \tag{3.103}
\end{equation*}
$$

since $\frac{[(m-1) l]}{[(m-1) l-1]} p_{\bar{s}}=n_{[s]}$. Consequently, evaluating this at $v=q$ yields $w w^{*}(v)=c n_{[s]}(q)$ for some non-zero $c$, since nominator and denominator each are evaluable at $q$ and since $n_{[s]}$ is evaluable.
5. Using (3.92) and (3.99) it is easy to see that

$$
n_{[t]} u^{*}=\frac{[m l]}{[m l-1]} p_{[t]} U_{m l-1} p_{[s]} \text { and } u n_{[t]}=\frac{[m l]}{[m l-1]} p_{[s]} U_{m l-1} p_{[t]}
$$

are both 0 at $q$. Moreover, if $m>1$, (3.103) and (3.99) give rise to

$$
\begin{aligned}
u^{*} n_{[s]} & =u^{*} \frac{[(m-1) l]}{[(m-1) l-1]} p_{\bar{s}}=p_{[t]} U_{m l-1} p_{\bar{s}} \frac{[(m-1) l]}{[(m-1) l-1]}=0, \\
n_{[s]} u & =\frac{[(m-1) l]}{[(m-1) l-1]} p_{\bar{s}} u=p_{[s]} U_{m l-1} p_{\bar{s}} \frac{[(m-1) l]}{[(m-1) l-1]}=0,
\end{aligned}
$$

since $p_{[t]} U_{m l-1} p_{\bar{s}}=0$ by Corollary 3.1.19, which is already true in $T L_{n}$.
Having talked about nilpotent elements, we can now prove (GW93, Theorem 2.2):

Theorem 3.2.28. 1. If $\lambda$ is critical, then $z_{\lambda}(q)$ is evaluable and minimal central as an idempotent in $T L_{n}(q)$. Furthermore, $z_{\lambda} T L_{n}(q) \cong M_{f_{\lambda}} \mathbb{C}$.
2. If $\lambda$ is non-critical, then $z_{\lambda}^{L}$ is evaluable. Moreover,
(a) $z_{\lambda}^{L} T L_{n} z_{\lambda}^{L}(q) \cong M_{f_{\lambda}^{L}} \mathbb{C}$, if $\lambda$ is to the left of the first critical line and
(b) $z_{\lambda}^{L} T L_{n} z_{\lambda}^{L}(q) \cong\left\{\left(\begin{array}{cc}A & B \\ 0 & A\end{array}\right), A, B \in M_{f_{\lambda}^{L}} \mathbb{C}\right\}$, if not.

Proof. 1. If $\lambda$ is critical, then $z_{\lambda}$ is evaluable by Proposition 3.2.26 and moreover it is known to be minimal central. By Lemma 3.2.3, we obtain that $\operatorname{dim} z_{\lambda} T L_{n}(q)=f_{\lambda}^{2}$, so if $U=\left\{u_{i j}\right\}$ is a minimal $z_{\lambda}$-system of evaluable matrix units enjoying Proposition 3.2.26, then $u_{i j} \mapsto E_{i j}$ implements an isomorphism $z_{\lambda} T L_{n}(q) \cong M_{f_{\lambda}} \mathbb{C}$, where $E_{i j}$ is the canonical basis of $M_{f_{\lambda}} \mathbb{C}$.
2. The same is true if $\lambda$ is non-critical but left of the first critical line, since in this case, one actually has $z_{\lambda}^{L}=z_{\lambda}$.
3. Finally if $\lambda$ is non-critical and between the $m$ th and the $m+1$ th critical lines for a $m \geq 1$, then $z_{\lambda}^{L}$ is evaluable by Proposition 3.2.26. In $T L_{n}$ the element $z_{\lambda}^{L}$ satisfies $=\sum_{t \in L(\lambda)} p_{t}+p_{\bar{t}}$, thus, with $\operatorname{dim} p_{t} T L_{n} p_{\bar{s}}=0$, we obtain that

$$
\operatorname{dim}_{\mathbb{C}(v)} z_{\lambda}^{L} T L_{n} z_{\lambda}^{L}=\sum_{t, s \in L(\lambda)} \operatorname{dim}\left(p_{t} T L_{n} p_{s}\right)+\operatorname{dim}\left(p_{\bar{t}} T L_{n} p_{\bar{s}}\right)=2\left(f_{\lambda}^{L}\right)^{2},
$$

implying with Lemma 3.2.3, that $\operatorname{dim}_{\mathbb{C}} z_{\lambda}^{L} T L_{n} z_{\lambda}^{L}(q)=2\left(f_{\lambda}^{L}\right)^{2}$. Now Proposition 3.2.26 states the existence of a $z_{\lambda}^{L}$-system $U=\left\{u_{i j}\right\}$ of evaluable matrix units of order $f_{\lambda}^{L}$ such that $U$ contains $u_{11}=p_{[t]}$, where $\bar{t} \in R(\lambda)$ is an evaluable regular tableau of shape $\lambda$ with critical subtableau of shape ( $m l-1$ ) (using Remark 3.2.25). Now $n_{[t]}=p_{[t]} U_{m l-1} p_{[t]}$ is non-zero and nilpotent of order 2 by Lemma 3.2.27, therefore it follows, that $N=\sum u_{i 1} n_{[t]} u_{1 i}$ is also non-zero and nilpotent of order 2, but commuting with all $u_{i j}$. Thus the subalgebra of $z_{\lambda}^{L} T L_{n} z_{\lambda}^{L}(q)$ generated by $N$ and the matrix units $u_{i j}$ is by Lemma 3.2.27 isomorphic to $M_{f_{\lambda}^{L}} \mathbb{C} \otimes \mathbb{C}[x] /\left(x^{2}\right)$, which is also isomorphic to $\left\{\left(\begin{array}{cc}A & B \\ 0 & A\end{array}\right), A ; B \in M_{f_{\lambda}^{L}} \mathbb{C}\right\}$. Since its dimension is $2\left(f_{\lambda}^{L}\right)^{2}$, it must already be all of $z_{\lambda}^{L} T L_{n} z_{\lambda}^{L}(q)$.

We now want describe the blocks, the radical and the semisimple quotient of $T L_{n}(q)$. We know that nilpotent central elements are always contained in the Jacobson radical. Before proving the main theorem, we start with:

Lemma 3.2.29. For $v, \mu \in[\lambda]$ adjacent and to the right of the first critical line

$$
z_{\mu}^{L} T L_{n} z_{v}^{L}(q) \operatorname{rad}\left(z_{v}^{L} T L_{n} z_{v}^{L}(q)\right)=\operatorname{rad}\left(z_{\mu}^{L} T L_{n} z_{\mu}^{L}(q)\right) z_{\mu}^{L} T L_{n} z_{\nu}^{L}(q)=\{0\}
$$

holds. Moreover,

$$
z_{\mu}^{L} T L_{n} z_{\nu}^{L} T L_{n} z_{\mu}^{L}(q)=\operatorname{rad}\left(z_{\mu}^{L} T L_{n} z_{\mu}^{L}(q)\right)
$$

Proof. Without loss of generality, $v$ is assumed to be to the left of $\mu$ and moreover let $v$ be also between the $m$ th and the $m+1$ th critical line for $m \geq 1$.

1. The proof of Theorem 3.2.28 explains why the radical of $z_{v} T L_{n} z_{v}(q)$ is generated by

$$
N=\sum u_{i 1} n_{[s]} u_{1 i}
$$

where $U=\left\{u_{i j}\right\}$ is a $z_{v}^{L}$-system of evaluable matrix units of order $f_{v}^{L}$ containing $p_{[s]}$, where $s \in L(v)$ is a regular tableau with critical subtableau of shape $(m l-1)$ and where $n_{[s]}=p_{[t]} U_{m l-1} p_{[s]}$. Remark 3.2.25 lets us assume that $s_{m(l+1)-1}$ is admissible for $s$ and that $t=s_{m(l+1)-1}(s)$ is regular in $R(v)$ with critical subtableau of shape $(m(l+1)-1)$, i.e. the situation is that in Figure 15 , such that $v=\operatorname{Shape}(s)=\operatorname{Shape}(t)$ and $\mu=\operatorname{Shape}(\bar{t})$. In particular, applying Lemma 3.2.27, its fourth part states that

$$
\begin{equation*}
\operatorname{rad}\left(z_{v}^{L} T L_{n} z_{v}^{L}(q)\right)=N z_{v}^{L} T L_{n} z_{v}^{L}(q) \tag{3.104}
\end{equation*}
$$

Let $V=\left\{v_{i j}\right\}$ be a $z_{\mu}^{L}$-system of evaluable matrix units of order $f_{\mu}^{L}$ containing $p_{[t]} . V$ exists, since $t$ is regular and in $L(\mu)$, since $t \in R(v)$, so we can assume that $v_{11}=p_{[t]}$ and that

$$
\begin{equation*}
v_{i i}=v_{i 1} p_{[t]} v_{1 i} \tag{3.105}
\end{equation*}
$$

With

$$
\begin{equation*}
z_{\mu}^{L} T L_{n} z_{v}^{L} N \subset \operatorname{span}\left(z_{\mu} T L_{n} z_{v} n_{[s]} T L_{n}\right) \tag{3.106}
\end{equation*}
$$

the equations (3.104), (3.106), (3.105) and the fact $z_{\mu}^{L}=\sum_{i} v_{i i}$ imply

$$
\begin{aligned}
z_{\mu}^{L} T L_{n} z_{v}^{L} \operatorname{rad}\left(z_{v}^{L} T L_{n} z_{v}^{L}\right)(q) & =z_{\mu}^{L} T L_{n} z_{v}^{L} N z_{v}^{L} T L_{n} z_{v}^{L}(q) \\
& \subset \operatorname{span}\left(z_{\mu} T L_{n} z_{v} n_{[s]} T L_{n} z_{v}^{L} T L_{n} z_{v}^{L}\right) \\
& \subset \operatorname{span}\left(\sum_{i} v_{i 1} p_{[t]} T L_{n} n_{[s]} T L_{n}(q)\right),
\end{aligned}
$$

which is $\{0\}$ by the fifth part of Lemma 3.2.27.
2. The same works also to show that $\operatorname{rad}\left(z_{\mu}^{L} T L_{n} z_{\mu}^{L}(q)\right) z_{\mu}^{L} T L_{n} z_{v}^{L}(q)=(0)$.

The second equation directly follows from the fact that $N_{[s]}$ generates the radical and from the fourth part of Lemma 3.2.27.

Now we are in a position to prove the following theorem, which is the main result Theorem 3.2.21 of this section:

Theorem 3.2.30 (Theorem 2.3, (GW93)). If $\lambda$ is non-critical, then $z_{[\lambda]}(q)$ is a minimal central idempotent in $T L_{n}(q)$. The radical of $z_{[\lambda]} T L_{n}(q)$ is nilpotent of order 3 and is spanned by the spaces $z_{\mu}^{L} T L_{n} z_{\nu}^{L}(q)$ for pairs of adjacent diagrams $\mu, v$ in the orbit $[\lambda]$ and by the algebras $\operatorname{rad}\left(z_{\mu}^{L} T L_{n} z_{\mu}^{L}(q)\right)$ for $\mu \in[\lambda]$. The maximal semisimple quotient of $z_{[\lambda]} T L_{n}(q)$ is isomorphic to $\bigoplus_{\mu \in[\lambda]} M_{f_{\mu}^{L}} \mathbb{C}$.

Proof. The element $z_{[\lambda]}:=\sum_{\mu \in[\lambda]} z_{\mu}^{L}=\sum_{\mu \in[\lambda]} z_{\mu}$ is an evaluable central idempotent.

1. We first calculate the radical of $z_{[\lambda]} T L_{n}(q)$. It is clear that

$$
\begin{equation*}
z_{[\lambda]} T L_{n}=\bigoplus_{\mu, v \in[\lambda]} z_{\mu}^{L} T L_{n} z_{v}^{L} \tag{3.107}
\end{equation*}
$$

(a) Assume that $\mu, v \in[\lambda]$ are adjacent. Without loss of generality $v$ is to the left of $\mu$, i.e. $v_{1} \leq \mu_{1}$, thus every path $t \in L(\mu)$ satisfies

$$
p_{[t]}=p_{t}+p_{\bar{t}} \text { and } \bar{t} \in \operatorname{Std}(v)
$$

In particular, for a path $s \in L(v), p_{s} T L_{n} p_{t}$ is one dimensional, so the dimension of $z_{\mu}^{L} T L_{n} z_{v}^{L} \subset T L_{n}$ is the number of those $\left(p_{t}, p_{s}\right)$ pairs, i.e. $f_{\mu}^{L} f_{v}^{L}$. However, Lemma 3.2.3 implies then $\operatorname{dim} z_{\mu}^{L} T L_{n} z_{v}^{L}(q)=f_{\mu}^{L} f_{v}^{L}$.
(b) If $\mu$ and $v$ are not adjacent, then Corollary 3.1.19 implies that $z_{\mu}^{L} T L_{n} z_{v}^{L}=$ $\{0\}$.

By the above two points, (3.107) refines as follows:

$$
\begin{equation*}
z_{[\lambda]} T L_{n}=\bigoplus_{\mu \in[\lambda]} z_{\mu}^{L} T L_{n} z_{\mu}^{L} \oplus \bigoplus_{\substack{\mu, v \in[\lambda] \\ \mu, v a d j a c e n t}} z_{\mu}^{L} T L_{n} z_{v}^{L} . \tag{3.108}
\end{equation*}
$$

Moreover, using Lemma 3.2.29 implies now that

$$
R_{[\lambda]}:=\bigoplus_{\substack{\mu, v[\lambda] \\ \mu, v \text { adjacent }}} z_{\mu}^{L} T L_{n} z_{\nu}^{L}(q) \oplus \bigoplus_{\mu \in[\lambda]} \operatorname{rad}\left(z_{\mu}^{L} T L_{n} z_{\mu}^{L}(q)\right)
$$

is a nilpotent ideal in $z_{[\lambda]} T L_{n}(q)$. But the quotient is semisimple:

$$
z_{[\lambda]} T L_{n}(q) / R_{[\lambda]}=\bigoplus_{\mu \in[\lambda]} z_{\mu}^{L} T L_{n} z_{\mu}^{L}(q) / \operatorname{rad}\left(z_{\mu}^{L} T L_{n} z_{\mu}^{L}(q)\right) \cong \bigoplus_{\mu \in[\lambda]} M_{f_{\mu}^{L}} \mathbb{C}
$$

so $R_{[\lambda]}$ is the radical of $z_{[\lambda]} T L_{n}(q)$.
2. Using Lemma 3.2.29 and the fact that $\operatorname{rad}\left(z_{\mu}^{L} T L_{n} z_{\mu}^{L}(q)\right)$ is of order 2, imply clearly that $R_{[\lambda]}$ is nilpotent of order 3 .
3. To show that $z_{\mu}^{L}$ is a minimal central idempotent modulo the radical, we observe first that $z_{\mu}^{L}$ is clearly idempotent and morever, that

- $z_{\mu}^{L}$ commutes with $z_{v}^{L} T L z_{v}^{L}$ if $\mu=v$,
- if $v \neq \mu$ are adjacent, then $z_{\mu}^{L} z_{\nu}^{L} T L_{n} z_{v}^{L}(q)$ is contained in the radical,
- If $v \neq \mu$ are not adjacent, the space $z_{\mu}^{L} z_{v}^{L} T L_{n} z_{v}^{L}(q)$ is zero and
- if $\eta \neq v$, then $z_{\mu}^{L} z_{v}^{L} T L_{n} z_{\eta}^{L}(q)$ is zero or always contained in the radical.

In particular, $z_{v}^{L}(q)$ commutes with $z_{[\lambda]}(q)$ modulo the radical, thus $z_{v}^{L}(q)$ is a minimal central idempotent modulo the radical.
4. It is left to show, that $z_{[\lambda]}(q)$ is minimal as an central idempotent in $T L_{n}(q)$. By the previous part every central idempotent in $z_{[\lambda]} T L_{n}(q)$ is a sum of some $z_{v}^{L}(q), v \in[\lambda]$ and an element in $R_{[\lambda]}$. However the reader may easily check, that no such element is central, if it is not $z_{[\lambda]}(q)$, by using that $R_{[\lambda]}$ is of order 3.

Having the main result proven, we finish this section by showing that the action defined in (2.24) is actually faithful for all $q \in \mathbb{C}^{\times}$.

Writing down that action from (2.24) by using the morphism $\phi_{1}$ from (2.7) implies that $U_{i}$ acts from the right by

$$
\pi_{q}: T L_{n}(q)^{\mathrm{op}} \rightarrow \operatorname{End}\left(V^{\otimes n}\right), U_{i} \mapsto \mathrm{id}_{V}^{\otimes(i-1)} \otimes T \otimes \mathrm{id}_{V}^{\otimes(n-i+1)},
$$

where $T \in \operatorname{End}(V \otimes V)$ is given by the rule

$$
v_{i} \otimes v_{j} \mapsto \begin{cases}v v_{i} \otimes v_{j}-v_{j} \otimes v_{i}, & \text { if } i<j \\ 0, & \text { if } i=j \\ v^{-1} v_{i} \otimes v_{j}-v_{j} \otimes v_{i}, & \text { if } i>j\end{cases}
$$

Now we can prove the following theorem, following (GW93, Theorem 2.4):

Theorem 3.2.31. The representation $\pi_{q}$ defined above of $T L_{n}(q)$ on $V^{\otimes n}$ is faithful.
If $q$ is not a $2 l$ th root of unity for $l=2, \ldots, n$, opposed to our global assumption of this section, the statement is just Proposition 2.3.10.

Proof. If $p \in T L_{n}$ is any idempotent evaluable at $q$, then $p$ is also evaluable at a neighborhood of $q$. If $\xi$ is in a deleted neighborhood of $q$, then $\pi_{\xi}$ is faithful, thus $\pi_{\xi}(p) \neq 0$, which implies with constancy of rank, that $\pi_{q}(p) \neq 0$, hence $\pi_{q}$ is faithful on the maximal semisimple quotient of $T L_{n}(q)$.

If $\lambda$ is non-critical, $t$ a regular tableau of shape $\lambda$ and $n_{[t]}(v)$ the non-zero nilpotent element in the two dimensional algebra $p_{[t]} T L_{n} p_{[t]}(q)$, then the proof of the first part of Lemma 3.2.27 shows, that there is an evaluable minimal idempotent $f$ in $T L_{N}$ for some $N>n$, such that $f n_{[t]} f(q)$ is a non-zero multiple of $f(q)$. The representation $\pi_{q}$ commutes with the usual embedding $T L_{n} \subset T L_{N}$ and $\operatorname{End}\left(V^{\otimes n}\right) \subset \operatorname{End}\left(V^{\otimes N}\right)$, so that $\pi_{q}(f) \neq 0$ implies $\pi_{q}\left(n_{[t]}\right) \neq 0$. But for for any nonzero element in $\operatorname{rad}\left(T L_{n}\right)$, the ideal generated by this element contains an element of the form $n_{[t]}$. In particular, $\pi_{q}$ is faithful on $\operatorname{rad}\left(T L_{n}\right)$.

Now the discussion of the minimal central idempotents modulo the radical as it is presented in (GW93) is finished. Although some of the proofs of this section were quite tiring, nevertheless we stress, that all statements not dealing with modules were here shown diagramatically. In particular, the construction of a sufficient number of evaluable idempotents, Proposition 3.2.26, did not use any nondiagramatic statements.

However, while proving the statements of Section 3.2.1, one could observe that elements implementing equivalences between path idempotents, seem to be constructed after a certain schema. For two paths $t$ and $s$ of same shape, we saw that the space $p_{s} T L_{n} p_{t}$ is one dimensional, thus there exists a basis $p_{s, t}$ indexed by pairs of tableaux $t, s$ of same shape, such that $p_{s, t} \in p_{s} T L_{n} p_{t}$. But it is also interesting to know the coefficents of this basis, a good aproach would be to construct this basis $p_{s, t}$ explicitly. This is done in the next section.

## An upper triangular relation

Let $p_{t}$ and $p_{s}$ be two path idempotents in $T L_{n}$ such that $t=s_{i}(s)$. If $q$ is not a root of unity, we saw in Proposition 3.2.10 that $p_{s}$ and $p_{t}$ are equivalent and moreover, that the elements $e, f$ implementing the equivalence are, up to multiplying with a scalar, the elements $p_{s} U_{i} p_{t}$ and $p_{t} U_{i} p_{s}$. It is also clear, that $p_{t} T L_{n} p_{s}$ being one dimensional is generated by $p_{t} U_{i} p_{s}$. Now if moreover $s_{j}$ is admissible for $s$ and $r=s_{j}(s)$, then one would obtain that $p_{t} U_{i} p_{s} U_{j} p_{r}$ and $p_{r} U_{j} p_{s} U_{i} p_{t}$ would implement, up to a scalar, an equivalence between $r$ and $t$ and secondly they would also generate the spaces $p_{r} T L_{n} p_{t}$ and $p_{t} T L_{n} p_{r}$. This leads to the idea to construct elements $p_{t, s} \in p_{t} T L_{n} p_{s}$ inductively and by choosing carefully the right coefficents, they would implement equivalences between $p_{t}$ and $p_{s}$. This is done in Section 4.1. It is clear, since $T L_{n}$ decomposes into a sum of one dimensional subspaces $p_{t} T L_{n} p_{s}$ for Shape $(t)=\operatorname{Shape}(s)$, that the elements $p_{t, s}$ form a basis, which will include the path idempotents $p_{t, t}=p_{t}$. However, we can actually calculate the coefficents involved to express $p_{t, s}$ in terms of another known basis, the cellular basis $\beta_{t, s}$ (see also Remark 2.2.6) used for the usual cellular structure on $T L_{n}$, defined in (GL96, Example 1.4). An example of this calculation is given in Section 4.2 and the general formula will be proven in Section 4.3. As a side result, we will be able to express the path idempotents $p_{t}$ in terms of the cellular basis $\beta_{t, s}$ with more or less explicit formulas, which is as far as we know a new formula.

The main results of this section are summarized by the following theorem (see Definition 4.1.8, Proposition 4.1.10 and Theorem 4.3.18):

Theorem. There exists a basis of $T L_{n}$ of elements $p_{t, s}$ indexed by pairs of tableaux $t$ and s of same shape such that

1. $p_{t, t}$ equals the path idempotent $p_{t}$,
2. the elements $p_{t, s}$ form a system of matrix units, i.e. $p_{t, s} p_{s, r}=p_{t, r}$ for all paths $t, s, r$ of same shape and
3. the basis $p_{t, s}$ is related by an upper triangular relation with respect to the dominance order to the cellular basis $\beta_{t, s}$ of (GL96).

Moreover for $u, v, t, s$ of same shape, the coefficent $c_{u, v}^{t, s} \in \operatorname{Std}(n)$ of $\beta_{u, v}$ in $p_{t, s}=$ $\sum_{u \unlhd t, v \unlhd s} c_{u, v}^{t, s} \beta_{t, s}$ is described by (4.43).

We start by defining explicitly the elements $p_{t, s}$.

## Section 4.1

A new basis
The starting point is to define $p_{t, s}$ for adjacent paths $t$ and $s$, actually they were already introduced implicitly as the elements $u$ and $w$ in the proof of 3.2.10:

Definition 4.1.1. Let $s \unlhd t$ be two paths of same shape, such that $s_{i}(s)=t$ and let $\emptyset \rightarrow \lambda^{(1)} \rightarrow \cdots \rightarrow \lambda^{(i)}$ be the maximal common subpath of $s$ and $t$, see Figure 16. Setting $d=\lambda_{1}^{(i)}-\lambda_{2}^{(i)}$, the elements $p_{t, s}$ and $p_{s, t}$ are defined by

$$
\begin{equation*}
p_{s, t}:=f_{s, t} p_{s} U_{i} p_{t} \text { and } p_{t, s}:=f_{t, s} p_{t} U_{i} p_{s} \tag{4.1}
\end{equation*}
$$

where $f_{s, t}=\frac{[d+1]}{[d]}$ and $f_{t, s}=\frac{[d+1]}{[d+2]}$.


Figure 16: Paths $s$ and $t$.

Remark 4.1.2. The proof of Proposition 3.2.10 justifies the choice of the coefficents $f_{t, s}$, since it implies that $p_{s, t} p_{t, s}=p_{s}$.

To construct elements $p_{t, s}$ inductively, we introduce the following notation:
Definition 4.1.3. Let $r$ and $t$ be two paths of same shape and $s_{i_{1}} \cdots s_{i_{k}} \in S_{n}$ be a minimal expression, such that

1. $\left(s_{i_{k}} \cdots s_{i_{1}}\right)(r)=t$ and
2. $r_{i_{m} \ldots i_{1}}:=\left(s_{i_{m}} \ldots s_{i_{1}}\right)(r)$ is always a standard tableau.

Then $s_{i_{1}} \cdots s_{i_{k}}$ is called a $(r, t)$-regular expression.
The following lemma is only needed in a relativly simple special case in this section, but it is also needed in Section 4.3.

Lemma 4.1.4. If $s_{1_{1}} \cdots s_{i_{k}}$ is a $(r, t)$-regular expression for two paths $r$ and $t$, then

$$
\begin{equation*}
p_{r} U_{i_{1}} \cdots U_{i_{k}} p_{t}=p_{r} U_{i_{1}} p_{r_{i_{1}}} U_{i_{2}} \cdots p_{r_{i_{k-1} \ldots i_{1}}} U_{i_{k}} p_{t} \tag{4.2}
\end{equation*}
$$

Proof. The first case to treat is the relativly easy case where $k=2$, i.e. where $t=s_{j} s_{i}(r)=r_{j i}$. Applying (3.11) proves this case:

$$
\begin{aligned}
p_{r} U_{i} U_{j} p_{t} & =p_{r}\left(p_{r}+p_{r_{i}}\right) U_{i} U_{j}\left(p_{r_{i}}+p_{r_{j i}}\right) p_{r_{j i}}=p_{r} U_{i}\left(p_{r}+p_{r_{i}}\right)\left(p_{r_{i}}+p_{r_{j i}}\right) U_{j} p_{r_{j i}} \\
& =p_{r} U_{i} p_{r_{i}} U_{j} p_{t}
\end{aligned}
$$

Now if $k \geq 3$, then let $J_{d}$ for $d \leq k$ be defined by

$$
J_{d}:=\left\{\left(j_{1}, \ldots, j_{h}\right) \in \mathbb{Z}^{h} ; 1 \leq j_{1}<\cdots<j_{h} \leq d\right\}
$$

This is the set of ordered tuples $\boldsymbol{j}$ of length at most $d$ in $\{1, \ldots, d\}$. For $\boldsymbol{j} \in J_{d}$ the tableau $r_{j}$ and the element $p_{j} \in T L_{n}$ are defined to be

$$
r_{j}:=s_{i_{j_{h}}} \ldots s_{i_{j_{1}}}(r) \in \operatorname{Tab}(\lambda) \text { and } p_{j}:= \begin{cases}p_{r_{j}}, & r_{j} \text { is standard }  \tag{4.3}\\ 0, & \text { otherwise }\end{cases}
$$

For example, if $\boldsymbol{j}=(1, \ldots, k) \in J_{k}$, then $r_{j}=r_{i_{k} \ldots i_{1}}=t$. Moreover, if $d<k$, and $\boldsymbol{j} \in J_{d} \subset J_{d+1}$, the tuple $\boldsymbol{j}^{1} \in J_{d+1}$ is defined to be

$$
\begin{equation*}
\boldsymbol{j}^{1}=\left(j_{1}, \ldots, j_{d}, d+1\right) \tag{4.4}
\end{equation*}
$$

which implies that $s_{i_{d+1}}\left(r_{j}\right)=r_{j^{1}}$ and moreover, that

$$
\begin{equation*}
p_{s_{i_{d+1}}\left(r_{j}\right)}=p_{\boldsymbol{j}^{1}} \tag{4.5}
\end{equation*}
$$

It is clear, that if $r_{j}$ is standard but $s_{i_{d+1}}\left(r_{j}\right)$ is not, then this means by using (3.7) that $p_{j} U_{i_{d+1}}=0$. Therefore, with Lemma 3.1.18, (4.4) and (4.5) the equation

$$
\begin{equation*}
\left(p_{j}+p_{j^{1}}\right) U_{i_{d+1}}=U_{i_{d+1}}\left(p_{j}+p_{j^{1}}\right) \tag{4.6}
\end{equation*}
$$

holds. Since $J_{d+1}$ is the disjoint union of $J_{d}$ and $\left\{\boldsymbol{j}^{1}: \boldsymbol{j} \in J_{d}\right\}$, (4.6) implies

$$
\sum_{j \in J_{d+1}} p_{j} U_{i_{d+1}}=\sum_{j \in J_{d}}\left(p_{\boldsymbol{j}}+p_{\boldsymbol{j}^{1}}\right) U_{i_{d+1}}=\sum_{j \in J_{d}} U_{i_{d+1}}\left(p_{\boldsymbol{j}}+p_{\boldsymbol{j}^{1}}\right)=U_{i_{d+1}} \sum_{j \in J_{d+1}} p_{\boldsymbol{j}}
$$

i.e. for $d<k$, we obtain

$$
\begin{equation*}
\sum_{j \in J_{d}} p_{j} U_{i_{d}}=U_{i_{d}} \sum_{j \in J_{d}} p_{j} \tag{4.7}
\end{equation*}
$$

We now show for $d \leq k$ the following equation by using induction:

$$
\begin{equation*}
p_{r} U_{i_{1}} \ldots U_{i_{d}}=p_{r} U_{i_{1}} \ldots U_{i_{d}} \sum_{j \in J_{d}} p_{j} \tag{4.8}
\end{equation*}
$$

Proof of the equation (4.8). The equation is clear for $d=1$, since then $\sum_{j \in J_{1}} p_{j}=$ $p_{r_{i_{1}}}+p_{r}$, so (4.8) for $d=1$ follows from Lemma 3.1.18. Now if $d>1$, then $J_{d-1} \subset J_{d}$ and the fact that the $p_{f}$ are idempotents imply together that

$$
\begin{equation*}
\sum_{j \in J_{d-1}} p_{j} \sum_{l \in J_{d}} p_{l}=\sum_{j \in J_{d-1}} p_{j} \tag{4.9}
\end{equation*}
$$

Then induction hypothesis for (4.8) and substituting (4.7) and (4.9) yield

$$
\begin{aligned}
p_{r} U_{i_{1}} \ldots U_{i_{d}} & =p_{r} U_{i_{1}} \ldots U_{i_{d-1}} \sum_{j \in J_{d-1}} p_{j} U_{i_{d}}=p_{r} U_{i_{1}} \ldots U_{i_{d-1}} \sum_{j \in J_{d-1}} p_{j} \sum_{l \in J_{d}} p_{l} U_{i_{d}} \\
& =p_{r} U_{i_{1}} \ldots U_{i_{d-1}} \sum_{j \in J_{d-1}} p_{j} U_{i_{d}} \sum_{l \in J_{d}} p_{l}=p_{r} U_{i_{1}} \ldots U_{i_{d-1}} U_{i_{d}} \sum_{l \in J_{d}} p_{l} .
\end{aligned}
$$

Applying induction hypothesis for (4.2) and using (4.8) allow us to deduce

$$
\begin{aligned}
p_{r} U_{i_{1}} \ldots U_{k} p_{r_{i_{k} \ldots i_{1}}} & =p_{r} U_{i_{1}} \ldots U_{k-1} \sum_{j \in J_{k-1}} p_{j} U_{k}\left(p_{r_{i_{k-1} \ldots i_{1}}}+p_{r_{i_{k} \ldots i_{1}}}\right) p_{r_{i_{k} \ldots i_{1}}} \\
& =p_{r} U_{i_{1}} \ldots U_{k-1} \sum_{j \in J_{k-1}} p_{j}\left(p_{r_{i_{k-1} \ldots i_{1}}}+p_{r_{i_{k} \ldots i_{1}}}\right) U_{k} p_{r_{i_{k} \ldots i_{1}}} \\
& =p_{r} U_{i_{1}} \ldots U_{k-1} p_{r_{i_{k-1} \ldots i_{1}}} U_{k} p_{r_{i_{k} \ldots i_{1}}} \\
& =p_{r} U_{i_{1}} p_{r_{i_{1}}} \ldots U_{i_{k-1}} p_{r_{i_{k-1} \cdots i_{1}}} U_{i_{k}} p_{r_{i_{k} \ldots i_{1}}},
\end{aligned}
$$

where we used $\sum_{j \in J_{k-1}} p_{j}\left(p_{r_{k-1} \cdots i_{1}}+p_{r_{k} \cdots i_{1}}\right)=p_{r_{i_{k-1} \cdots i_{1}}}$.
These ( $r, t$ )-regular expressions are actually braid-avoiding, i.e. they do not contain subwords of the form $s_{i} s_{i \pm 1} s_{i} \in S_{n}$. This will be used later.

Lemma 4.1.5. Any $(r, t)$-regular expression $s_{i_{1}} \cdots s_{i_{k}}$ is braid-avoiding.
Proof. There are eight possibilities for a tableau where to place the numbers $i, i+$ $1, i+2$ indicated by Figure 17. We leave it to the reader to check that $s_{i} s_{i+1} s_{i}$ is not


Figure 17: Eight possibilities.
admissible in all cases. The same works for $s_{i+1} s_{i} s_{i+1}$.
Remark 4.1.6. 1. It is known, that two minimal braid-avoiding expressions in $S_{n}$ can be obtained from each other by only appling the relation

$$
\begin{equation*}
s_{i} s_{j}=s_{j} s_{i} \text { for }|i-j|>1 \tag{4.10}
\end{equation*}
$$

This is for example shown in (Ste96, Proposition 2.1). However, if $r$ is a standard tableau, such that $s_{j}(r)$ and $s_{i} s_{j}(r)$ are still standard, i.e. such that $s_{j}$ and $s_{i} s_{j}$ are admissible for the path $r$, then also $s_{i}$ and $s_{j} s_{i}$ are admissible for $r$. In particular, the set of $(r, t)$-regular expressions is preserved under (4.10).
2. A $(r, t)$-regular expression $s_{i_{1}} \cdots s_{i_{k}}$ is also minimal as an expression in $S_{n}$.

Since (4.10) corresponds to the relation (2.6) in $T L_{n}$, we obtain the following consequence:

Corollary 4.1.7. For two $(r, t)$-regular expressions $s_{i_{1}} \cdots s_{i_{k}}=s_{j_{1}} \cdots s_{j_{k}}$

$$
\begin{equation*}
p_{r, r_{1}} p_{r_{i_{1}}, r_{i_{2} i_{1}}} \cdots p_{r_{i_{k-1} \cdots i_{1}}, r_{i_{k} \cdots i_{1}}}=p_{r, r_{j_{1}}} p_{r_{j_{1}}, r_{j_{2} j_{1}}} \cdots p_{r_{j_{k-1} \cdots j_{1}}, r_{j_{k} \cdots j_{1}}} \tag{4.11}
\end{equation*}
$$

holds, where $p_{r_{j_{h} \ldots j_{1}}, r_{j_{h+1} \ldots j_{1}}}$ is defined for $r_{j_{h} \ldots j_{1}}$ and $r_{j_{h+1} \ldots j_{1}}$ in Definition 4.1.1.
Proof. We consider first the special case, where $k=2$ and $t=s_{i} s_{j}(r)=s_{j} s_{i}(r)$ for two admissible $s_{j}, s_{i}$, such that $|i-j|>1$. Let $w$ and $s$ be defined by $w=s_{j}(t)=s_{i}(r)$ and $s=s_{i}(t)=s_{j}(r)$; the situation is depicted in Figure 18. This implies


Figure 18: Paths $r$ and $t$ and two different $(r, t)$-regular expressions.

$$
f_{r, w}=f_{s, t} \text { and } f_{r, s}=f_{w, t}
$$

and moreover,

$$
f_{r, w} f_{w, t}=f_{s, t} f_{r, s}=f_{r, s} f_{s, t}
$$

thus we obtain

$$
\begin{align*}
p_{r, w} p_{w, t} & =f_{r, w} f_{w, t} p_{r} U_{i} p_{w} U_{j} p_{t}=f_{r, s} f_{s, t} p_{r} U_{i} U_{j} p_{t}=f_{r, s} f_{s, t} p_{r} U_{j} U_{i} p_{t} \\
& =f_{r, s} f_{s, t} p_{r} U_{j} p_{s} U_{i} p_{t}=p_{r, s} p_{s, t} \tag{4.12}
\end{align*}
$$

Now the general case follows directly from the special case, i.e. from (4.12), since by Lemma 4.1.5 every $(r, t)$-regular expression is braid-avoiding and therefore two $(r, t)$-regular expressions can be obtained from each other by applying only the relation (4.10).

By using Corollary 4.1.7, the following definition is well-defined:
Definition 4.1.8. Let $r$ and $t$ be two paths ending in $\lambda \in \operatorname{Par}_{2}(n)$. Let $s_{i_{1}} \cdots s_{i_{k}}$ be a $(r, t)$-regular expression. If $r=t$, then we define $p_{r, r}$ to be the path idempotent $p_{r}$. If $r \neq t$, then we define $p_{r, t}$ by

$$
p_{r, t}=p_{r, r_{i_{k}, \ldots i_{1}}}=f_{r, t} p_{r} U_{i_{1}} p_{r_{i_{1}}} U_{2} \cdots p_{r_{i_{k-1} \cdots i_{1}}} U_{i_{k}} p_{r_{i_{k}, \ldots i_{1}}},
$$

where we set $f_{r, t}=f_{r, r_{i_{1}}} \cdots f_{r_{i_{k-1} \ldots i_{1}}, r_{i_{k}, i_{1}}}$.
As a first property, Definition 4.1.8 and Remark 4.1.2 have the following consequences:

Corollary 4.1.9. 1. For two paths $s, t$ of same shape, $p_{s, t} p_{t, s}=p_{s}$ holds.
2. The set of elements $p_{t, s}$ for $t, s \in \operatorname{Std}(n)$ of same shape is a basis of $T L_{n}$.

Proof. Remark 4.1.2 implies the first statement if $t$ and $s$ are adjacent, which implies with the inductive construction in Definition 4.1.8 the first statement for all $t$ and $s$ of same shape. Now to prove the second statement, the first important information is that $p_{t, s}$ is non-zero for all paths of same shape, since $p_{s}=p_{s, t} p_{t, s}$ is non-zero. Therefore, if we had a linear relation of the form

$$
\sum_{\substack{t, s \in \operatorname{Std}(n) \\ \operatorname{Shape}(t)=\operatorname{Shape}(s)}} c_{t, s} p_{t, s}=0
$$

then multiplying from the left by $p_{\tau}$ and from the right by $p_{\sigma}$ would imply that $c_{\tau, \sigma} p_{\tau, \sigma}=0$, hence $c_{\tau, \sigma}=0$ for all $\tau, \sigma \in \operatorname{Std}(n)$ of same shape. In particular the elements $p_{t, s}$ are linearly independant. Moreover, knowing that the elements $p_{t}$ form a complete set of idempotents, $T L_{n}$ decomposes as

$$
T L_{n}=\bigoplus_{\substack{t, s \in \operatorname{Std}(n)}} p_{t} T L_{n} p_{s}=\bigoplus_{\substack{t, s \in \operatorname{Std}(n) \\ \text { Shape(t)=Shape}(s)}} p_{t} T L_{n} p_{s}
$$

where we used Corollary 3.1 .26 for the last equation. But since $p_{t, s} \in p_{t} T L_{n} p_{s}$, which is one dimensional by Corollary 3.2.11, this implies that the elements $p_{t, s}$ generate $T L_{n}$ as a vector space, thus they form a basis.

The equation $p_{t, s} p_{s, t}=p_{t}$ actually has a stronger consequence, which means that the basis $p_{t, s}$ is a set of matrix units:

Proposition 4.1.10. For three paths $w, t$ and $r$ of same shape, $p_{w, t} p_{t, r}=p_{w, r}$ holds.

Proof. Let $s=s_{i_{1}} \cdots s_{i_{k}}$ be a $(w, t)$-regular expression and $s^{\prime}=s_{i_{k+1}} \cdots s_{i_{k+h}}$ a $(t, r)$ regular expression. Let $m \leq k+h$ be the length of a ( $w, r$ )-regular expression. We proceed by induction over $k+h-m$.

1. If $m=k+h$, we see that $s s^{\prime}$ must be a ( $w, r$ )-regular expression: It satisfies

$$
\begin{align*}
& \left(s_{i_{k+h}} \cdots s_{i_{1}}\right)(w)=r \text { and }  \tag{4.13}\\
& \quad\left(s_{i_{m}} \cdots s_{i_{1}}\right)(w) \text { is standard for all } 1 \leq m \leq k+h . \tag{4.14}
\end{align*}
$$

Since $k+h=m, s s^{\prime}$ must also be minimal with that property.
2. Assume that $m<k+h$. Although $s s^{\prime}$ is not minimal, it satisfies (4.13) and (4.14). By the deletion condition (cf. (Hum90, Section 1.7)), there exist $1 \leq m<m^{\prime} \leq k+h$ such that

$$
\begin{equation*}
s_{i_{m+1}} \cdots s_{i_{m^{\prime}}}=s_{i_{m}} \cdots s_{i_{m^{\prime}-1}} \tag{4.15}
\end{equation*}
$$

It is clear that $m, m^{\prime} \leq k$ and $m, m^{\prime}>k$ are impossible, since the expressions for $s$ and $s^{\prime}$ are also minimal expressions in $S_{n}$ (see Remark 4.1.6), thus $m \leq k<m^{\prime}$. Without loss of generality, we can assume that $m \leq k$ is maximal and $m^{\prime}>k$ is minimal. But then (4.15) describes also a $\left(t_{1}, t_{2}\right)$ regular expression, where

$$
t_{1}=\left(s_{i_{m}} \cdots s_{i_{1}}(w) \text { and } t_{2}=\left(s_{i_{m^{\prime}}} \cdots s_{i_{1}}\right)(w) .\right.
$$

We use the notation $w_{i_{d} \cdots i_{1}}$ for $s_{i_{d}} \cdots s_{i_{1}}(w)$. Since $p_{t_{1}, t_{2}}$ is independant of the choice of a $\left(t_{1}, t_{2}\right)$-regular expression, (4.15) implies

$$
\begin{aligned}
& p_{t_{1}, t_{2}}=p_{w_{i_{m} \ldots i_{1}}, w_{i_{m+1} \ldots i_{1}}} \cdots p_{w_{i_{m^{\prime}-1} \ldots i_{1}}, w_{i_{m^{\prime}} \ldots i_{1}}} \\
& =p_{w_{i_{m} \ldots i_{1}}, w_{i_{m i} \ldots \ldots i_{1}}} \cdots p_{w_{i_{m^{\prime}-2} \ldots i_{m} i_{m} \ldots i_{1}}, w_{i_{m^{\prime}-1} \ldots i_{m} i_{m} \ldots i_{1}}} \\
& =p_{w_{i_{m} \ldots i_{1}}, w_{i_{m-1} \ldots i_{1}}} \cdots p_{w_{i_{m^{\prime}-2} \ldots} \ldots \ldots i_{1}, \ldots, w_{i_{m^{\prime}-1} \ldots \widehat{i_{m}} \ldots i_{1}}},
\end{aligned}
$$

where $\widehat{i_{m}}$ means that $i_{m}$ is omitted. But this implies

$$
\begin{align*}
& p_{w, t} p_{t, r}=\left(p_{w, w_{i_{1}}} \cdots p_{w_{i_{m-1} \ldots i_{1}}, w_{i_{m} \ldots i_{1}}}\right)\left(p_{w_{i_{m} \ldots i_{1}}, w_{i_{m+1} \ldots i_{1}}} \cdots p_{w_{i_{m^{\prime}-1} \ldots i_{1}}, w_{i_{m^{\prime}}, \ldots i_{1}}}\right) \\
& \cdot\left(p_{w_{i_{m^{\prime}}, i_{1}}, w_{i_{m^{\prime}+1} \ldots i_{1}}} \cdots p_{w_{i_{k+h-1} \ldots i_{1}}, w_{i_{k+h} \ldots i_{1}}}\right) \\
& =(p_{w, w_{i_{1}}} \cdots \underbrace{p_{w_{i_{m-1} \ldots i_{1}}, w_{i_{m} \ldots i_{1}}} p_{w_{i_{m} \ldots i_{1}}, w_{i_{m-1}, \ldots i_{1}}}}_{p_{w_{i_{m-1}} \ldots i_{1}}} \cdots p_{w_{i_{m^{\prime}-2}, \overparen{i_{m}} \ldots i_{1}}, w_{i_{i^{\prime}-1}, \ldots, \bar{m}_{m} \ldots i_{1}}}) \\
& \cdot\left(p_{w_{i_{m^{\prime}} \cdots i_{1}}, w_{i_{m^{\prime}+1} \ldots i_{1}}} \cdots p_{\left.w_{i_{k+h-1} \cdots i_{1}, w_{i_{k+h} \ldots i_{1}}}\right)}\right) \tag{4.16}
\end{align*}
$$

where we also used

$$
s_{i_{1}} \cdots s_{i_{k+h}}=s_{i_{1}} \cdots \widehat{s_{i_{m}}} \cdots \widehat{s_{i_{m^{\prime}}}} \cdots s_{i_{h+k}}
$$

for the last equation. So, (4.16) implies that $s_{i_{1}} \cdots \widehat{s_{i_{m}}} \cdots \widehat{s_{i_{m^{\prime}}}} \cdots s_{i_{h+k}}$ satisfies (4.13) and (4.14) but is of length $h+k-2$. Using induction hypothesis gives the desired result.

A good idea would be now to calculate some $p_{s, t}$ to get an impression how the look like. Therefore we are going to express the $p_{s, t}$ in terms of an "easy" diagramatic basis, namely that which has been defined in Section 2.2, see Definition 2.2.4. In Section 4.2 this is exemplarily done in the case $n=4$.

Section 4.2
The case $n=4$
We woul like to find out how $\left\{p_{t, s}, t, s \in \operatorname{Std}(\lambda), \lambda \in \operatorname{Par}_{2}(n)\right.$ is related to the ceullular basis of $T L_{n}$ consisting of arc diagrams. For $n=4$ the branching graph for the algebras $T L_{1} \subset \cdots \subset T L_{4}$ is indicated in the first picture of Figure 19.





Figure 19: The branching graph for $T L_{4}$ with the paths $t \in \operatorname{Std}(4)$.

We have three partitions of 4 , namely the partitions $\lambda=(4), \mu=(3,1)$ and $v=(2,2)$ with their corresponding tableaux $s_{1} \in \operatorname{Std}(\lambda), s_{2}, s_{3}, s_{4} \in \operatorname{Std}(\mu)$ and $s_{5}, s_{6} \in \operatorname{Std}(v)$. The transition matrix between the basis

$$
\left\{p_{11}, p_{22}, p_{23}, p_{24}, p_{32}, p_{33}, p_{34}, p_{42}, p_{43}, p_{44}, p_{55}, p_{56}, p_{65}, p_{66}\right\}
$$

ant the basis

$$
\left\{\beta_{11}, \beta_{22}, \beta_{23}, \beta_{24}, \beta_{32}, \beta_{33}, \beta_{34}, p_{42}, \beta_{43}, \beta_{44}, \beta_{55}, \beta_{56}, \beta_{65}, \beta_{66}\right\}
$$

can be obtained by using Definition 3.1.7 and Definition 4.1.8. Doing so one ob-
tains the following matrix $A$ :

$$
A=\left(\begin{array}{cccccccccccccc}
1 & -\frac{[3]}{[4]} & \frac{[2]}{4]} & -\frac{1}{[4]} & \frac{[2]}{[4]} & -\frac{[2]^{2}}{[4]} & \frac{[2]}{[4]} & -\frac{1}{[4]} & \frac{[2]}{[4]} & -\frac{[3]}{[4]} & \frac{[2]}{[3][4]} & \frac{1-[3]}{[3][4]} & \frac{1-[3]}{[3][4]} & \frac{[2][3]+[2]}{[3]] 4]}  \tag{4.17}\\
0 & \frac{[3]}{[4]} & -\frac{[2]}{[4]} & \frac{1}{[4]} & -\frac{[2]}{[4]} & \frac{[2]^{2}}{[3][4]} & -\frac{[2]}{[3][4]} & \frac{1}{[4]} & -\frac{[2]}{[3][4]} & \frac{1}{[3][4]} & -\frac{[2]}{[3][4]} & \frac{[3]-1}{[3][4]} & \frac{[3]-1}{[3][4]} & -\frac{[2][3]+[2]}{[3][4]} \\
0 & 0 & \frac{[2]}{4]} & -\frac{1}{[4]} & 0 & -\frac{[2]^{2}}{[3][4]} & \frac{[2]}{[3][4]} & 0 & \frac{[2]}{[3][4]} & -\frac{1}{[3][4]} & \frac{[2]}{[3][4]} & -\frac{1}{[3][4]} & -\frac{[2]^{2}}{[4]} & \frac{[2]}{[2]} \\
0 & 0 & 0 & \frac{1}{[4]} & 0 & 0 & -\frac{[2]}{[3][4]} & 0 & 0 & \frac{1}{[3][4]} & 0 & \frac{1}{[3][4]} & 0 & -\frac{2}{[3][4]} \\
0 & 0 & 0 & 0 & 1 & -\frac{[2]}{[3]} & \frac{1}{[3]} & \frac{1}{[2]} & \frac{1}{[3]} & -\frac{1}{[2][3]} & \frac{1}{[3]} & -[2] & -\frac{1}{[2][3]} & 1 \\
0 & 0 & 0 & 0 & 0 & \frac{[2]}{[3]} & -\frac{1}{[3]} & 0 & -\frac{1}{[3]} & \frac{1}{[2][3]} & -\frac{1}{[3]} & \frac{1}{[2][3]} & \frac{1}{[2][3]} & -\frac{1}{[2]^{2}[3]} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{[3]} & 0 & 0 & -\frac{1}{[2] 3} & 0 & -\frac{1}{[2][3]} & 0 & \frac{1}{[2]^{2}[3]} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{[3]}{[2]} & -1 & \frac{1}{[2]} & 0 & 0 & \frac{1}{[2]} & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{[2]} & 0 & 0 & -\frac{1}{[2]} & \frac{1}{[2]^{2}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{[2]} & 0 & 0 & 0 & -\frac{1}{\left[22^{2}\right.} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{[3]} & -\frac{1}{[2][3]} & -\frac{1}{\frac{1}{22][3]}} & \frac{1}{\left.[2]^{2} 23\right]} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{[2][3]} & 0 & -\frac{1}{\left[22^{2}[3]\right.} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{[2]} & -\frac{1}{[2]^{2}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{[2]^{2}}
\end{array}\right)
$$

Therefore our hope is to show, that the set

$$
\left\{p_{s, t}, s, t \in \operatorname{Std}(\lambda), \lambda \in \operatorname{Par}_{2}(n)\right\}
$$

is related via an upper triangular relation to the cellular basis

$$
\left\{\beta_{s, t}, s, t \in \operatorname{Std}(\lambda), \lambda \in \operatorname{Par}_{2}(n)\right\} .
$$

This will be done in Section 4.3.

## Section 4.3

Relating with a cellular basis
Our first goal is to express the path idempotents $p_{t}$ in terms of the cellular basis $\beta_{t, s}$. The coefficents involved will depend on the possibilities to construct the path $t$ by "replacing" $\epsilon_{j}(t)=1$ by -1 . To do so, we first need to find the right indices $j$, where this will be done.

Definition 4.3.1. Let $t=\left((1)=\lambda^{(1)} \rightarrow \cdots \rightarrow \lambda^{(n)}\right)$ be a path ending in $\lambda^{(n)}=\lambda$ with $d=\lambda_{1}-\lambda_{2}$. For $h=1, \ldots, d$ the path $r_{h}^{t}$ is defined to be the minimal subpath

$$
\begin{equation*}
r_{h}=r_{h}^{t}=\lambda^{(1)} \rightarrow \cdots \rightarrow \lambda^{\left(j_{h}\right)} \tag{4.18}
\end{equation*}
$$

such that $\lambda_{1}^{(k)}-\lambda_{2}^{(k)} \geq h$ for all $k \geq j_{h}$.
Remark 4.3.2. 1. By the choice of $r_{h}$ we see that $\lambda^{\left(j_{h}\right)}=\left(\lambda_{1}^{\left(j_{h}-1\right)}+1, \lambda_{2}^{\left(j_{h}-1\right)}\right)$. Moreover, $r_{h}$ exists for $h=1, \ldots, d$ since $\lambda_{1}-\lambda_{2} \geq h$. In the branching graph, this means that after the endpoint of $r_{h}$ the path $t$ is weakly to the right of the line defined by all partitions $\mu$ with $\mu_{1}-\mu_{2}=h$, see also Figure 20 .


Figure 20: Paths $r_{h}, r_{h+1}$ and $\#_{h}(t)$.
2. The path $r_{k}$ is a subpath of $r_{h}$ if and only if $k \leq h$ and moreover the number $j_{1}, \ldots, j_{d}$ are exactly the labels of the vertical lines on the top edge in the diagram $\beta_{t, \text {. }}$ (see Definition 2.2.4).
3. Clearly the subpath $r_{h}^{t}$ satisfies $\epsilon_{j_{h}}\left(r_{h}\right)=1=\epsilon_{j_{h}}(t)$.

The index $j_{h}$ will be the right place to replace $\epsilon_{j_{h}}=1$ by -1 :
Definition 4.3.3. For $d \geq 2$ and $h \in\{1, \ldots, d-1\}$ the path $\#_{h}(t) \in \operatorname{Std}\left(\lambda_{1}-1, \lambda_{2}+1\right)$ is defined by:

$$
\epsilon_{i}\left(\#_{h}(t)\right)= \begin{cases}\epsilon_{i}(t), & i \neq j_{h+1}  \tag{4.19}\\ -1, & i=j_{h+1}\end{cases}
$$

It is clear that $\#_{h}(t) \triangleleft t$ for all $h \in\{1, \ldots, d-1\}$. An example of $\#_{h}(t)$ with subpaths $r_{h}$ and $r_{h+1}$ is given in the second picture of Figure 20. When $p_{r}$ is described in the basis $\beta_{t, s}$, the coefficent of $\beta_{t, s}$ will depend on the ways to construct $t$ and $s$ out of the maximal path $t^{(n)} \in \operatorname{Std}(n)$. However, to prove the later statements, we will need a slight generalization of the previous definition:

Definition 4.3.4. For $\lambda \in \operatorname{Par}_{2}(n)$ with $d=\lambda_{1}-\lambda_{2}$ and $\mu \in \operatorname{Par}_{2}(d)$, the partition $\lambda \# \mu$ is said to be the partition $\left(\lambda_{1}-\mu_{2}, \lambda_{2}+\mu_{2}\right)$. Moreover, for $t \in \operatorname{Std}(\lambda)$ and $r \in \operatorname{Std}(\mu)$, the tableau $t \# r \in \operatorname{Std}(\lambda \# \mu)$ is defined as follows:

$$
\epsilon_{i}(t \# r)= \begin{cases}\epsilon_{i}(t), & \text { if } i \neq j_{h} \forall h=1, \ldots, d,  \tag{4.20}\\ \epsilon_{h}(r), & \text { if } i=j_{h}\end{cases}
$$

Remark 4.3.5. 1. If $r$ has only one number $k$ in the second row corresponding to a horizontal line $(h, k)$ in $\beta_{r, \text {, }}$, then we obtain that $t \# r=\#_{h}(r)$. This was the example in Figure 20.
2. $\mu_{1}+\mu_{2}=\lambda_{1}-\lambda_{2}$ implies that

$$
\left(\lambda_{1}-\mu_{2}\right)-\left(\lambda_{2}+\mu_{2}\right)=\lambda_{1}-\lambda_{2}-2 \mu_{2}=\mu_{1}+\mu_{2}-2 \mu_{2}=\mu_{1}-\mu_{2} \geq 0
$$

thus $\lambda \# \mu$ is a well-defined two-row partition of $n$.
3. It is clear, that if $r \neq t^{(d)}$, i.e. if $\mu_{2} \neq 0$, then we obtain that $t \# r \triangleleft t$.

The reason why to define this composition of tableaux is the following quite obvious fact, which is clear by the construction:

Lemma 4.3.6. If $t \in \operatorname{Std}(\lambda)$ and $r \in \operatorname{Std}(\mu)$ where $\mu \in \operatorname{Par}_{2}\left(\lambda_{1}-\lambda_{2}\right)$, then

$$
\begin{equation*}
\beta_{t, \cdot} \cdot \beta_{r, \cdot}=\beta_{t \# r, \cdot} \tag{4.21}
\end{equation*}
$$

The following lemma will be needed to prove the coefficent formula: It will be proven by using induction and appling Definition 3.1.1.
Lemma 4.3.7. If $x \in T L_{n, d}$, then


Proof. Since $p_{d}=p_{d} \cdot\left(p_{d-1} \sqcup 1\right)$, it suffices to show the equation:


Clearly (4.22) holds for $d=1$, therefore $d$ is assumed to be greater than 1 . Then Definition 3.1.1 and the induction hypothesis imply together that




$$
=\left.\sum_{j=1}^{d}(-1)^{d-j} \frac{[j]}{[d]}\right|_{d-1} ^{\cdots}
$$

We actually need a certain consequence of the previous lemma as well. Therefore, if $r$ is a path, let $I_{r}^{-}$be the set of indices $i$ such that $\epsilon_{i}(r)=-1$, i.e. $I_{r}^{-}$consists of the numbers in the second row of the tableau $r$. Let $\#^{-1}(r)$ be the set of pairs $(j, \sigma)$ such that $j \in I_{r}^{-}$and $\#_{j} \sigma=r$. Then Lemma 4.3.6 has the following consequence:

Corollary 4.3.8. If $x=\sum_{\sigma \in \operatorname{St}(\lambda)} \gamma_{\sigma} \beta_{\sigma, \text {, }}$ is an element of $T L_{n, d}$ where $d=\lambda_{1}-\lambda_{2}$, then
holds, where

$$
\begin{equation*}
\gamma_{w}=\sum_{\left(j, \sigma^{+}\right) \in \oiint^{-1}(w)} \gamma_{\sigma}(-1)^{d+j} \frac{[j]}{[d]} . \tag{4.23}
\end{equation*}
$$

Proof. If $x=\sum_{\sigma \in \operatorname{Std}(\lambda)} \gamma_{\sigma} \beta_{\sigma, \text {, }}$, then

$$
\begin{equation*}
x \sqcup 1=\sum_{\sigma \in \operatorname{Std}(\lambda)} \gamma_{\sigma} \beta_{\sigma^{+},} \in T L_{n+1, d+1} . \tag{4.24}
\end{equation*}
$$

Now for $j=1, \ldots, d$ let $\xi_{j} \in \operatorname{Std}((d+1,1))$ be the tableau of shape $(d+1,1)$ with $j+1$ in the second row, so that $\beta_{\xi_{j} \text {; }}$, is given by

$$
\beta_{\xi_{j},}=\left.\left.\right|^{\cdots}\right|^{j} \bigcup^{\cdots} .
$$

By using Lemma 4.3.6 and the first part of Remark 4.3.5, the equation

$$
\begin{equation*}
\beta_{\sigma, \cdot} \cdot \beta_{\xi_{j},}=\beta_{\#_{j}(\sigma)} \tag{4.25}
\end{equation*}
$$

holds, and therefore (4.24) and (4.25) imply together

$$
\begin{aligned}
& =\sum_{j=1}^{d}(-1)^{d-j} \frac{[j]}{[d]} \sum_{\sigma \in \operatorname{Std}(\lambda)} \gamma_{\sigma} \beta_{\#_{j}\left(\sigma^{+}\right),,},
\end{aligned}
$$

which implies (4.23).
Now having shown a first technical lemma, it is time to define the coefficents involved to express $p_{r}$ in terms of basis elements $\beta_{t, s}$ :


Figure 21: A path $r$ and the sequence $r^{(0)}, r^{(1)}, r^{(2)}, r^{(3)}, r^{(4)}, r^{(5)}=r$.

Definition 4.3.9. Let $w \in \operatorname{Std}(\lambda)$ be a tableau of shape $\lambda \in \operatorname{Par}_{2}(n)$.

1. $J_{w}$ is defined as the set of sequences $\left(w^{(0)}, \ldots, w^{\left(\lambda_{2}\right)}\right)$ of tableaux such that
(a) $w^{(k+1)}=\#_{h} w^{(k)}$ for some $h$ and
(b) $w^{\left(\lambda_{2}\right)}=w$ and $w^{(0)}=t^{(n)}$.
2. A sequence $\mathbf{w} \in J_{w}$ is said to dominate another sequence $\mathbf{v} \in J_{v}$, denoted by $\mathbf{w} \unlhd \mathbf{v}^{\prime}$, if and only if $w^{(k)} \unlhd v^{(k)}$ for all $1 \leq k \leq \lambda_{2}$. If $J_{r}^{r} \subset J_{r}$ denotes the singleton containing the unique minimal sequence $\mathbf{r} \in J_{r}$ with respect to $\unlhd$, then the set $J_{w}^{r}$ is defined to be the subset of $\mathbf{w} \in J_{w}$ such that $\mathbf{w} \unlhd \mathbf{r} \in J_{r}^{r}$.
3. For a sequence $\mathbf{w}$ and two successive $w^{(k-1)}$ and $w^{(k)}$ let $\mu$ be the shape of their maximal common subpath. Then $d_{\mathbf{w}}^{k}$ is defined by $d_{\mathbf{w}}^{k}=\mu_{1}-\mu_{2}$.
4. Finally the coefficient $C_{w}^{r}$ is defined to be

$$
\begin{equation*}
C_{w}^{r}=\sum_{\mathbf{w} \in J_{w}^{r}} C_{\mathbf{w}} \text { with } C_{\mathbf{w}}=\prod_{k=1}^{d}(-1)^{d_{\mathbf{w}}^{k}}\left[d_{\mathbf{w}}^{k}\right] . \tag{4.26}
\end{equation*}
$$

These coefficents $C_{w}^{r}$ will be the key ingredient to describe $p_{r}$ in terms of $\beta_{t, s}$.

## Example 4.3.10.

1. An example of a path $r$ and the only element $\mathbf{r} \in J_{r}^{r}$ is given in Figure 21.
2. In this particular example, the maximal common subpath of $r^{(0)}$ and $r^{(1)}$ is of shape $(2,0)$. In particular $d_{\mathbf{r}}^{1}=2$. Similar one sees $d_{\mathbf{r}}^{2}=4, d_{\mathbf{r}}^{3}=3, d_{\mathbf{r}}^{4}=5$ and $d_{\mathbf{r}}^{5}=4$, which implies

$$
C_{\mathbf{r}}=(-1)^{2}[2] \cdot(-1)^{4}[4] \cdot(-1)^{3}[3] \cdot(-1)^{5}[5] \cdot(-1)^{4}[4]=[2][3][4]^{2}[5] .
$$

Remark 4.3.11. 1. A sequence of tableaux $\mathbf{w} \in J_{w}$ could also be expressed by a sequence of indices $i_{1}, \ldots, i_{k}$ such that $\#_{i_{k}} \ldots \#_{i_{1}}=w^{(k)}$. In particular, $\mathbf{r} \in J_{r}^{r}$ corresponds to the unique sequence $j_{1}, \ldots, j_{d}$ such that $j_{1}<\cdots<j_{d}$.
2. Clearly a sequence $\mathbf{w}$ is ordered with respect to the dominance order: $(n)=$ $w^{(0)} \triangleright \cdots \triangleright w^{(k)}=w$.
3. Since $J_{r}^{r}$ consists only of the element $\mathbf{r} \in J_{\mathbf{r}}, C_{r}^{r}$ is given by

$$
C_{r}^{r}=C_{\mathbf{r}}=\prod_{k=1}^{d}(-1)^{d_{\mathbf{r}}^{k}}\left[d_{\mathbf{r}}^{k}\right] .
$$

If $p_{r}$ is a path idempotent expressed by $p_{d}=f_{r} x p_{d} \tilde{x}$, then $x$ can assumed to have through-degree $d$. In particular, if $x$ is written in terms of $\beta_{w,}$, we can write down a first formula to describe the coefficents of the $\beta_{w, \cdot}$.

Proposition 4.3.12. Let $r \in \operatorname{Std}(\lambda)$ and let $p_{r}=f_{r} x p_{d} \tilde{x}$, such that $x$ is of throughdegree $d=\lambda_{1}-\lambda_{2}$. Then

$$
\begin{equation*}
x=\sum_{\substack{w \in \operatorname{sid}(x) \\ w \leq r}} \gamma_{w}^{x} \beta_{w, r} \text { with } \gamma_{w}^{x}=\frac{C_{w}^{r}}{C_{r}^{r}} \tag{4.27}
\end{equation*}
$$

holds, where $C_{w}^{r}$ is defined in (4.26).
Proof. It is easy to see that $\beta_{\sigma, \cdot}, \sigma \in \operatorname{Std}(\lambda)$ is a basis of $T L_{n, d}$, therefore, since $x$ has through-degree $d, x$ can be expressed by

$$
\begin{equation*}
x=\sum_{\sigma \in \operatorname{Stt}(\lambda)} \gamma_{\sigma}^{x} \beta_{\sigma, \cdot} \tag{4.28}
\end{equation*}
$$

Now there are two cases:

1. If $\epsilon_{n}(r)=-$, i.e. if $r=r^{\prime-}$, then

Lemma 4.3.7 states that

which implies with (4.29)

$$
\begin{equation*}
\left.x=\sum_{j=1}^{d+1}(-1)^{d+1-j} \frac{[j]}{[d+1]}| |_{\ldots .}^{\mid \cdots \cdot}|\underbrace{j}_{j}|{ }_{j}^{y}|\quad| \cdots \right\rvert\, \tag{4.30}
\end{equation*}
$$

Howoever, by induction hypothesis $y$ satisfies

$$
\begin{align*}
y & =\sum_{\substack{w \in \operatorname{Std}\left(\left(\lambda_{1}, \lambda_{2}\right)\right) \\
w \leq r^{\prime}}} \gamma_{w}^{y} \beta_{w,}, \\
\gamma_{w}^{y} & =C_{\mathbf{r}^{\prime}}^{-1} \sum_{\mathbf{w} \in J_{w}^{\prime}} C_{\mathbf{w}} \text { and } C_{\mathbf{w}}=\prod_{k=1}^{d+1}(-1)^{d_{\mathbf{w}}^{k}}\left[d_{\mathbf{w}}^{k}\right] . \tag{4.31}
\end{align*}
$$

which implies with (4.28) and Corollary 4.3 .8 that

$$
\begin{equation*}
x=\sum_{\sigma \in \operatorname{Std}\left(\left(\lambda_{1}, \lambda_{2}+1\right)\right)} \gamma_{\sigma}^{x} \beta_{\sigma, \cdot} \text { and } \gamma_{\sigma}^{x}=\sum_{\left(j, w^{+}\right) \in \#^{-1}(\sigma)} \gamma_{w}^{y}(-1)^{d+1+j} \frac{[j]}{[d+1]} \tag{4.32}
\end{equation*}
$$

Since $[d+1]=d_{\mathbf{r}}^{d}$ implies $(-1)^{d+1}[d+1] C_{\mathbf{r}^{\prime}}=C_{\mathbf{r}}$ and therefore

$$
\begin{equation*}
(-1)^{d+1}[d+1] C_{r^{\prime}}^{r^{\prime}}=C_{r}^{r} \tag{4.33}
\end{equation*}
$$

the previous equations (4.31) and (4.33) yield together

$$
\begin{align*}
\gamma_{w}^{y}(-1)^{d+1+j} \frac{[j]}{[d+1]} & =(-1)^{j} \frac{[j]}{C_{r}^{r}} \sum_{\mathbf{w} \in J_{w}^{r^{\prime}}} \prod_{k=1}^{d+1}(-1)^{d_{\mathbf{w}}^{k}}\left[d_{\mathbf{w}}^{k}\right] \\
& =\left(C_{r}^{r}\right)^{-1} \sum_{\mathbf{w} \in J_{w}^{J^{\prime}}}(-1)^{j}[j] \prod_{k=1}^{d+1}(-1)^{d_{\mathbf{w}}^{k}}\left[d_{\mathbf{w}}^{k}\right] \\
& =\left(C_{r}^{r}\right)^{-1} \sum_{\substack{\mathbf{v} \in J_{r}^{r} \\
d \mathbf{v}_{\mathbf{v}}^{d}=j}} C_{\mathbf{v}}, \tag{4.34}
\end{align*}
$$

where (4.26) was used for the last equality. Since $\left(j, w^{+}\right) \in \#^{-1}(\sigma)$ if and only if for every sequence $\mathbf{w} \in J_{w}^{r^{\prime}}$, we can add $\#_{j}\left(w^{(d)^{+}}\right)$to obtain a sequence $\hat{\mathbf{w}} \in J_{\sigma}^{r}$, and since in this case $d_{\hat{\mathbf{w}}}^{d+1}=[j]$ holds, (4.32) and (4.34) together imply that

$$
\begin{equation*}
\gamma_{\sigma}^{x}=\sum_{\substack{\left(j, w^{+}\right) \in \#^{-1}(\sigma)}} \gamma_{w}^{y}(-1)^{d+1+j} \frac{[j]}{[d+1]}=\sum_{\substack{\left(j, w^{+}\right) \in \#^{-1}(\sigma) \\ w \leq r^{\prime}}}\left(C_{r}^{r}\right)^{-1} \sum_{\substack{\mathbf{v} \in J_{\sigma}^{r} \\ d_{\mathbf{v}}^{d}=j}} C_{\mathbf{v}}=\left(C_{r}^{r}\right)^{-1} \sum_{\mathbf{v} \in J_{\sigma}} C_{\mathbf{v}} \tag{4.35}
\end{equation*}
$$

Moreover, by induction hypothesis we can assume that $\gamma_{w}^{y}=0$, whenever $w \nexists r^{\prime}$. Therefore, since $\sigma \nexists r$ implies for $\left(j, w^{+}\right) \in \#^{-1}(\sigma)$ that $w \nexists r^{\prime}$, the implication

$$
\sigma \nexists r \Rightarrow \gamma_{\sigma}^{x}=0
$$

holds. In particular, (4.27) follows in this case from (4.35) and (??)
2. On the other hand, $\operatorname{if} \epsilon_{n}(r)=+1$, i.e. if $r=r^{\prime+}$, then it follows that

$$
p_{r^{\prime}}=f_{r^{\prime}} y p_{d-1} \tilde{y} \text { such that } f_{r}=f_{r^{\prime}} \text { and } x=y \sqcup 1 .
$$

Moreover, induction hypothesis assures $y=\sum_{\substack{w \in \operatorname{Std}(\mu) \\ w \unlhd r^{\prime}}} \gamma_{w}^{y} \beta_{w,}$, where $\mu=\left(\lambda_{1}-\right.$ $1, \lambda_{2}$ ), which implies

$$
x=y \sqcup 1=\sum_{\substack{w \in \operatorname{Std}(\mu) \\ w \unlhd r^{\prime}}} \gamma_{w}^{y} \beta_{w^{+}, \cdot},
$$

hence $\gamma_{\sigma}^{x}=\gamma_{\sigma^{\prime}}^{y}$. Therefore, since induction hypothesis ensures $\gamma_{\sigma^{\prime}}^{y}=0$ for $\sigma^{\prime} \nexists r^{\prime}$, which implies $\gamma_{\sigma}^{x}=0$ for $\sigma \nexists r$, this also shows (4.27) in this case.

Now it is time to show the first result of this section, namely the coefficent formula for $p_{r}$ for $\beta_{t, s}$ where $t$ and $s$ are of same shape as $r$. To do so, we fix some more notation: For $\lambda \in \operatorname{Par}_{2}(n), M^{\triangleleft \lambda}$ will denote the subspace of $T L_{n}$ spanned by $\beta_{u, v}$ such that $\operatorname{Shape}(u)=\operatorname{Shape}(v) \triangleleft \lambda ; M^{\triangleleft \lambda}$ is a two-sided ideal in $T L_{n}$.

Corollary 4.3.13. If $r$ is a path ending in $\lambda \in \operatorname{Par}_{2}(n)$, then

$$
p_{r}=p_{r, r} \equiv \sum_{(u, w) \unlhd(r, r)} c_{u, w}^{r} \beta_{u, w} \quad\left(\bmod M^{\triangleleft \lambda}\right)
$$

where the sum runs over all pairs $(u, w)$ such that $u \unlhd r$ and $w \unlhd r$. Moreover

$$
\begin{equation*}
c_{u, w}^{r}=f_{r} \frac{C_{u}^{r} C_{w}^{r}}{C_{r}^{r} C_{r}^{r}} \tag{4.36}
\end{equation*}
$$

holds, where $C_{u}^{r}$ is defined in (4.26) and $f_{r}$ in Definition 3.1.5.
Proof. If $p_{r}=f_{r} x p_{d} \tilde{x}$, then $x$ can assumed to have through-degree $d$, so it is possible to apply Proposition 4.3.12, which yields

$$
\begin{equation*}
x=\sum_{\substack{w \in \operatorname{Std}(\lambda) \\ w \leq r}} \gamma_{w}^{x} \beta_{w, \cdot} \text { with } \gamma_{w}^{x}=\frac{C_{w}^{r}}{C_{r}^{r}} . \tag{4.37}
\end{equation*}
$$

Furthermore, $p_{d}$ can be expressed by

$$
\begin{equation*}
p_{d}=\sum_{\substack{\tau, \pi \in \operatorname{Std}(d) \\ \operatorname{Shape}(\tau)=\operatorname{Shape}(\pi)}} \gamma_{\tau, \pi}^{d} \beta_{\tau,} \beta \cdot \beta \cdot \pi \tag{4.38}
\end{equation*}
$$

since $\beta_{\tau, \pi}=\beta_{\tau, \beta} \beta_{\cdot, \pi}$ is a basis of $T L_{d}$. Applying (4.37), (4.38) and Lemma 4.3.6 now implies

$$
\begin{align*}
p_{r} & =f_{r} x p_{d} \tilde{x}=f_{r}\left(\sum_{\sigma \in \operatorname{Std}(\lambda)} \gamma_{\sigma}^{x} \beta_{\sigma, \cdot}\right)\left(\sum_{\substack{\tau, \pi \in \operatorname{sta}(d) \\
\text { Shape }(\tau)=\operatorname{Shape}(\pi)}} \gamma_{\tau, \pi}^{d} \beta_{\tau, \beta} \beta_{\cdot, \pi}\right)\left(\sum_{\xi \in \operatorname{Std}(\lambda)} \gamma_{\sigma}^{x} \beta_{\cdot, \xi}\right) \\
& =f_{r} \sum_{\sigma, \tau, \pi, \xi} \gamma_{\sigma}^{x} \gamma_{\tau, \pi}^{d} \gamma_{\xi}^{x} \beta_{\sigma,,} \beta_{\tau, \beta} \beta_{\cdot, \pi} \beta_{\cdot, \xi}=f_{r} \sum_{\sigma, \tau, \pi, \xi} \gamma_{\sigma}^{x} \gamma_{\tau, \pi}^{d} \gamma_{\xi}^{x} \beta_{\sigma \# \tau, \xi \# \pi \cdot,} \tag{4.39}
\end{align*}
$$

where the last two sums range over all $\sigma, \xi \in \operatorname{Std}(\lambda)$ and $\tau, \pi \in \operatorname{Std}(d)$, such that $\tau$ and $\pi$ are of same shape. Since Shape $(\sigma \# \tau) \triangleleft \operatorname{Shape}(\sigma)=\lambda$ for $\tau \neq t^{(d)}$ and since the coefficent $\gamma_{t^{(d)}, t^{(d)}}$ of 1 in $p_{r}$ is 1 , (4.39) actually says that

$$
\begin{equation*}
c_{\sigma, \xi}^{r}=f_{r} \gamma_{\sigma}^{x} \gamma_{\xi}^{x} . \tag{4.40}
\end{equation*}
$$

But then the result follows from (4.39), (4.40) and (4.37).
We obtain an important consequence:
Corollary 4.3.14. If $r$ is a path, then the coefficent of $\beta_{r, r}$ in $p_{r}$ is given by $c_{r, r}^{r}=f_{r}$, where $f_{r}$ is defined in Definition 3.1.5.

Example 4.3.15. We reconsider the example of Section 4.2.

1. We want to use (4.26) to calculate the coefficent $\gamma_{s_{3}, s_{3}}^{s_{2}, s_{2}}=\gamma_{33}^{22}$ of $\beta_{s_{3}, s_{3}}=\beta_{33}$ in $p_{s_{2}, s_{2}}=p_{22}$. The set $J_{s_{2}}=J_{s_{2}}^{s_{2}}$ has only one element, namely the sequence $\mathbf{S}_{\mathbf{2}}=\left(s_{1}, s_{2}\right)$, which implies by applying (4.26) that

$$
\begin{equation*}
C_{2}^{2}=C_{s_{2}}^{s_{2}}=C_{\mathbf{s}_{2}}=(-1)^{3}[3], \tag{4.41}
\end{equation*}
$$

since the maximal common subpath of $s_{1}$ and $s_{2}$ is of shape $(3,0)$. The set $J_{s_{3}}$ also consists only of one element, namely the sequence $\mathbf{s}_{3}=\left(s_{1}, s_{3}\right)$ and moreover, $\mathbf{s}_{3} \triangleleft \mathbf{S}_{\mathbf{2}}$ since $s_{3} \triangleleft s_{2}$, which lets us obtain

$$
\begin{equation*}
C_{3}^{2}=C_{s_{3}}^{s_{2}}=(-1)^{2}[2], \tag{4.42}
\end{equation*}
$$

since the maximal common subpath is of shape $(2,0)$.
Now applying (4.36) yields

$$
\gamma_{33}^{22}=f_{s_{2}} \frac{C_{3}^{2} C_{3}^{2}}{C_{2}^{2} C_{2}^{2}}=\frac{[3]}{[4]} \frac{[2]^{2}}{[3]^{2}}=\frac{[2]^{2}}{[3][4]},
$$

but this is the calculated entry corresponding to $\beta_{3,3}$ in $p_{2}$ given in (4.17).
2. Also the set $J_{s_{4}}^{s_{4}}$ has only one element, namely the sequence $\mathbf{s}_{\mathbf{4}}=\left(s_{1}, s_{4}\right)$ and since the maximal common subpath of $s_{4}$ and $s_{1}$ is of shape $(1,0)$, the equation

$$
C_{4}^{2}=C_{s_{4}}^{s_{2}}=(-1)[1]=-1
$$

holds. Furthermore, we see that

$$
\gamma_{44}^{22}=f_{s_{2}} \frac{C_{4}^{2} C_{4}^{2}}{C_{2}^{2} C_{2}^{2}}=\frac{[3]}{[4]} \frac{1}{[3]^{2}}=\frac{1}{[3][4]},
$$

which is the coefficent of $\beta_{44}$ in $p_{22}$ calculated in (4.17).
3. The previous two examples imply that

$$
\gamma_{43}^{22}=f_{s_{2}} \frac{C_{4}^{2} C_{3}^{2}}{C_{2}^{2} C_{2}^{2}}=\frac{[3]}{[4]} \frac{(-1)[2]}{[3]^{2}}=-\frac{[2]}{[3][4]}
$$

But this is again the coefficent of $\beta_{4,3}$ in $p_{2}$.
4. On the other hand $J_{s_{5}}=J_{5}$ has two elements, namely the sequences $\mathbf{s}_{1}=$ $\left(s_{1}, s_{2}, s_{5}\right)$ and $\mathbf{s}_{\mathbf{5}}=\left(s_{1}, s_{3}, s_{5}\right)$, however, $\mathbf{s}_{\mathbf{5}} \triangleleft \mathbf{S}_{\mathbf{5}_{1}}$ since $s_{3} \triangleleft s_{2}$, thus

$$
C_{5}^{5}=C_{s_{5}}^{s_{5}}=(-1)^{2}[2](-1)[1]=[2],
$$

since the maximal common subpath of $s_{3}$ and $s_{1}$ is of shape $(2,0)$ and that of $s_{3}$ and $s_{5}$ is of shape $(2,1)$. Also $J_{6}=J_{s_{6}}$ consists of two elements, namely the sequences $\mathbf{s}_{61}=\left(s_{1}, s_{2} s_{6}\right)$ and $\mathbf{s}_{\mathbf{6} 2}=\left(s_{1}, s_{4}, s_{6}\right)$, but $\mathbf{s}_{\mathbf{6} 1} \nsubseteq \mathbf{s}_{51}$ since $s_{2} \nexists s_{3}$. In particular, $C_{6}^{5}$ consists only of the element $\mathbf{s}_{62}$ and therefore $C_{6}^{5}$ is given by

$$
C_{6}^{5}=C_{s_{6}}^{s_{5}}=(-1)(-1)=1
$$

since the maximal common subpath of $s_{1}$ and $s_{4}$ is of shape $(1,0)$ and the maximal common subpath of $s_{4}$ and $s_{6}$ is of shape $(2,1)$. So together we obtain

$$
\gamma_{66}^{55}=\gamma_{S_{6}, s_{6}}^{s_{5}, s_{5}}=f_{s_{5}} \frac{C_{6}^{5} C_{6}^{5}}{C_{5}^{5} C_{5}^{5}}=\frac{1}{[3]} \frac{1}{[2]^{2}}=\frac{1}{[2]^{2}[3]}
$$

which is the coefficent of $\beta_{66}$ in $p_{55}$ given in (4.17).
Remark 4.3.16. Note that the above example is not a particular good example, since $n=4$ does not include non-comparible paths of same shape and moreover all sets $J_{s}^{t}$ consist only of one element. However already for $n=5$, we obtain 42 basis elements $\beta_{t, s}$, which is tedious to calcalute in detail...

The next step is to prove, that also the elements $p_{s, t}$ satisfy an relation of the form

$$
p_{s, t} \equiv \sum_{(u, v) \unlhd(s, t)} c_{u, v}^{s, t} \beta_{u, v} \quad\left(\bmod M^{\triangleleft \lambda}\right)
$$

where $s, t \in \operatorname{Std}(\lambda)$. This is now quite easy by exploiting the properties of the path idempotents. We fix some more notation:

Definition 4.3.17. . If $s, t$ are two paths of same shape such that $s=s_{i}(t)$, then the coefficent $g_{s, t}$ is defined by

$$
g_{s, t}= \begin{cases}f_{s, t}, & \text { if } s \triangleleft t, \\ f_{t, s}, & \text { if } s \triangleright t .\end{cases}
$$

Inductively, for two arbitrary paths $r$ and $t$ of same shape and an $(r, t)$-regular expression $s_{i_{i}} \ldots s_{i_{k}}$, the coefficent $g_{r, t}$ is defined to be

$$
g_{r, t}=g_{r, r_{i_{1}}} g_{r_{i_{1}}, r_{i_{2} i_{1}}} \cdots g_{r_{i_{k-1} \ldots i_{1}}, r_{i_{k} \cdots, i_{1}}}
$$

A priori the coefficent $g_{r, t}$ depends on the choice of the $(r, t)$-regular expression. But the following theorem, proving a coefficent formula for $p_{t, s}$, implicitly also shows that $g_{r, t}$ is independant of this choice.

Theorem 4.3.18. If $r, t \in \operatorname{Std}(\lambda)$ for $\lambda \in \operatorname{Par}_{2}(n)$, then

$$
p_{r, t} \equiv \sum_{(u, w) \unlhd(r, t)} c_{u, w}^{r, t} \beta_{u, w} \quad\left(\bmod M^{\triangleleft \lambda}\right)
$$

and moreover for $u, w \in \operatorname{Std}(\lambda)$, the coefficent $c_{u, w}^{r, t}$ is given by

$$
\begin{equation*}
c_{u, w}^{r, t}=f_{r} g_{r, t} \frac{C_{u}^{r} C_{v}^{t}}{C_{r}^{r} C_{t}^{t}}, \tag{4.43}
\end{equation*}
$$

where $g_{r, t}$ is defined in Definition 4.3.17 and $C_{u}^{r}$ in (4.26).
Proof. Let $s_{i_{1}} \cdots s_{i_{k}}$ be a ( $s, t$ )-regular expression. To shorten formulas, we will abbreviate the path $r_{i_{j} \ldots i_{1}}$, if it arises as an index, by the symbol ( $j$ ). With this notation in mind, we also write

$$
p_{(j)}=f_{(j)} x_{(j)} p_{d} \tilde{x}_{(j)} \text { and } p_{r}=p_{(0)}=x_{(0)} p_{d} \tilde{x}_{(0)}
$$

By using Definition 4.1.8, $p_{r, t}$ is expressable by

$$
\begin{align*}
p_{r, t} & =f_{(0),(k)} p_{(0)} \prod_{j=1}^{k} U_{i_{j}} p_{(j)}=f_{(0),(k)} f_{(0)} x_{(0)} p_{d} \tilde{x}_{(0)} \prod_{j=1}^{k} f_{(j)} U_{i_{j}} x_{(j)} p_{d} \tilde{x}_{(j)} \\
& =f_{(0),(k)} f_{(0)} x_{(0)} p_{d} \tilde{x}_{(0)}\left(\prod_{j=1}^{k-1} f_{(j)} U_{i_{j}} x_{(j)} p_{d} \tilde{x}_{(j)}\right) f_{(k)} U_{i_{k}} x_{(k)} p_{d} \tilde{x}_{(k)} \\
& =f_{(0),(k)} f_{(0)} x_{(0)} p_{d} y_{k} p_{d} \tilde{x}_{(k)}, \tag{4.44}
\end{align*}
$$

where

$$
y_{k}=\tilde{x}_{(0)}\left(\prod_{j=1}^{k-1} f_{(j)} U_{i_{j}} x_{(j)} p_{d} \tilde{x}_{(j)}\right) f_{(k)} U_{i_{k}} x_{(k)}
$$

is an element of $T L_{d}$. In particular the statement of the theorem follows then by (4.27), (4.44) and the following equation

$$
\begin{equation*}
f_{(0),(k)} f_{(0)} x_{(0)} p_{d} y_{k} p_{d} \tilde{x}_{(k)}=g_{(0),(k)} f_{(0)} x_{(0)} p_{d} \tilde{x}_{(k)} \tag{4.45}
\end{equation*}
$$

Proof of of the equation (4.45). If $k=1$, i.e. if $r=s_{i_{1}} t$ and $r \triangleleft t$, then an easy diagramatic argument, which was done similarly in Section 3, for example in the proof of Proposition 3.2.10, shows that

$$
\begin{equation*}
f_{(1)} p_{d} \tilde{x}_{(0)} U_{i} x_{(1)} p_{d}=p_{d} \tag{4.46}
\end{equation*}
$$

But this implies that

$$
\begin{align*}
f_{(0),(1)} f_{(0)} x_{(0)} p_{d} y_{1} p_{d} \tilde{x}_{(1)} & =f_{(0),(1)} f_{(0)} x_{(0)} p_{d} \tilde{x}_{(0)} f_{(1)} U_{i_{1}} x_{(1)} p_{d} \tilde{x}_{(1)} \\
& =f_{(0),(1)} f_{(0)} x_{(0)} p_{d} \tilde{x}_{(1)}=g_{(0),(1)} f_{(0)} x_{(0)} p_{d} \tilde{x}_{(1)} . \tag{4.47}
\end{align*}
$$

If $r \triangleright t$, then similarly to (4.46), one can show that

$$
f_{(0)} p_{d} \tilde{x}_{(0)} U_{i_{1}} x_{(1)} p_{d}=p_{d}
$$

implying

$$
\begin{align*}
f_{(0),(1)} f_{(0)} x_{(0)} p_{d} y_{1} p_{d} \tilde{x}_{(1)} & =f_{(0),(1)} f_{(0)} x_{(0)} p_{d} \tilde{x}_{(0)} f_{(1)} U_{i_{1}} x_{(1)} p_{d} \tilde{x}_{(1)} \\
& =f_{(0),(1)} f_{(1)} x_{(0)} p_{d} \tilde{x}_{(1)}=f_{(1),(0)} f_{(0)} x_{(0)} p_{d} \tilde{x}_{(1)} \\
& =g_{(0),(1)} f_{(0)} x_{(0)} p_{d} \tilde{x}_{(1)} \tag{4.48}
\end{align*}
$$

Now (4.47) and (4.48) cover the case $k=1$.
On the other hand, if $k>1$, then applying (4.45) for $y_{k-1}$ and for the case $k=1$ yields

$$
\begin{aligned}
f_{(0),(k)} f_{(0)} x_{(0)} p_{d} y_{k} p_{d} \tilde{x}_{(k)} & =f_{(k-1),(k)} f_{(0),(k-1)} f_{(0)} x_{(0)} p_{d} y_{k-1} p_{d} \tilde{x}_{(k-1)} f_{(k)} U_{i_{k}} x_{(k)} p_{d} \tilde{x}_{(k)} \\
& =f_{(k-1),(k)} g_{(0),(k-1)} f_{(0)} x_{(0)} p_{d} \tilde{x}_{(k-1)} f_{(k)} U_{i_{k}} x_{(k)} p_{d} \tilde{x}_{(k)} \\
& =g_{(k-1),(k)} g_{(0),(k-1)} f_{(0)} x_{(0)} p_{d} \tilde{x}_{(k)}=g_{(0),(k)} f_{(0)} x_{(0)} p_{d} \tilde{x}_{(k)} .
\end{aligned}
$$

Thus we have shown (4.45).
The previous result generalizes Corollary 4.3.13. We stress the following implicit statement:

Corollary 4.3.19. The cellular basis $\beta_{t, s}$ and the basis $p_{t, s}$ are in an upper triangular relation with respect to the dominance order.

Example 4.3.20. We finish by discussing the example in Section 4.2. By Definition 4.3.17 the coefficent $g_{42}$ for the paths $s_{t}$ and $s_{2}$ of Figure 19 is given by

$$
f_{3,4} f_{2,3}=\frac{[3]}{[2]} \frac{[2]}{1}=[3]
$$

Multiplying with $f_{4}=\frac{1}{[2]}$ and using (4.41) and (4.42) gives that

$$
\gamma_{43}^{42}=f_{4} g_{42} \frac{C_{4}^{4} C_{3}^{2}}{C_{4}^{4} C_{2}^{2}}=\frac{1}{[2]}[3] \frac{[2]}{-[3]}-1,
$$

which is the coefficent of $\beta_{43}$ in $p_{42}$ we calculated in (4.17).

## References

[CH15] B. Cooper and M. Hogancamp. An exceptional collection for Khovanov homology. Algebr. Geom. Topol., 15(5):2659-2707, 2015.
[DJ86] R. Dipper and G. James. Representations of Hecke algebras of general linear groups. Proc. London Math. Soc. (3), 52(1):20-52, 1986.
[Du95] J. Du. A note on quantized Weyl reciprocity at roots of unity. Algebra Colloq., 2(4):363-372, 1995.
[GL96] J. J. Graham and G. I. Lehrer. Cellular algebras. Invent. Math., 123(1):1-34, 1996.
[GW93] F. M. Goodman and H. Wenzl. The Temperley-Lieb algebra at roots of unity. Pacific J. Math., 161(2):307-334, 1993.
[HPK91] H. Haahr Andersen, P. Polo, and W. Kexin. Representations of quantum algebras. Invent. Math., 104(1):1-59, 1991.
[HST15a] H. Haahr Andersen, C. Stroppel, and D. Tubbenhauer. Cellular structures using U_q-tilting modules. ArXiv e-prints, March 2015, 1503.00224. To appear in Pacific J. Math.
[HST15b] H. Haahr Andersen, C. Stroppel, and D. Tubbenhauer. Semisimplicity of Hecke and (walled) Brauer algebras. ArXiv e-prints, July 2015, 1507.07676. To appear in the Journal of the Australian Mathematical Society.
[Hum90] J. E. Humphreys. Reflection groups and Coxeter groups, volume 29 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1990.
[Hä99] Martin Härterich. Murphy bases of generalized Temperley-Lieb algebras. Arch. Math. (Basel), 72(5):337-345, 1999.
[Jim86] M. Jimbo. A $q$-analogue of $U(\mathfrak{g l}(N+1))$, Hecke algebra, and the YangBaxter equation. Lett. Math. Phys., 11(3):247-252, 1986.
[Jon85] V. F. R. Jones. A polynomial invariant for knots via von Neumann algebras. Bull. Amer. Math. Soc. (N.S.), 12(1):103-111, 011985.
[KL94] L. H. Kauffman and S. L. Lins. Temperley-Lieb recoupling theory and invariants of 3-manifolds, volume 134 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 1994.
[Kna07] A. W. Knapp. Advanced algebra. Cornerstones. Birkhäuser Boston, Inc., Boston, MA, 2007. Along with a companion volume it Basic algebra.
[KP96] Hanspeter Kraft and Claudio Procesi. Classical invariant theory, a primer. http://jones.math.unibas.ch/~kraft/docs/ primernew. pdf, 1996.
[LZ10] G. I. Lehrer and R. B. Zhang. A Temperley-Lieb analogue for the BMW algebra. In Representation theory of algebraic groups and quantum groups, volume 284 of Progr. Math., pages 155-190. Birkhäuser/Springer, New York, 2010.
[Mur95] G. E. Murphy. The representations of Hecke algebras of type $A_{n}$. J. Algebra, 173(1):97-121, 1995.
[Ros88] M. Rosso. Finite dimensional representations of the quantum analog of the enveloping algebra of a complex simple lie algebra. Communications in Mathematical Physics, 117(4):581-593, Dec 1988.
[Ste96] J. R. Stembridge. On the fully commutative elements of Coxeter groups. J. Algebraic Combin., 5(4):353-385, 1996.
[Wee12] T. Weelinck. Representation Theory of the Temperley-Lieb Algebra and its connections with the Hecke Algebra. Bachelor thesis, Universiteit van Amsterdam, July 2012. https://esc.fnwi.uva.nl/ thesis/centraal/files/f1637901874.pdf.
[Wen87] H. Wenzl. On sequences of projections. C. R. Math. Rep. Acad. Sci. Canada, 9(1):5-9, 1987.
[Wen88] H. Wenzl. Hecke algebras of type $A_{n}$ and subfactors. Invent. Math., 92(2):349-383, 1988.
[Wey39] H. Weyl. The Classical Groups. Their Invariants and Representations. Princeton University Press, Princeton, NJ, 1939.

